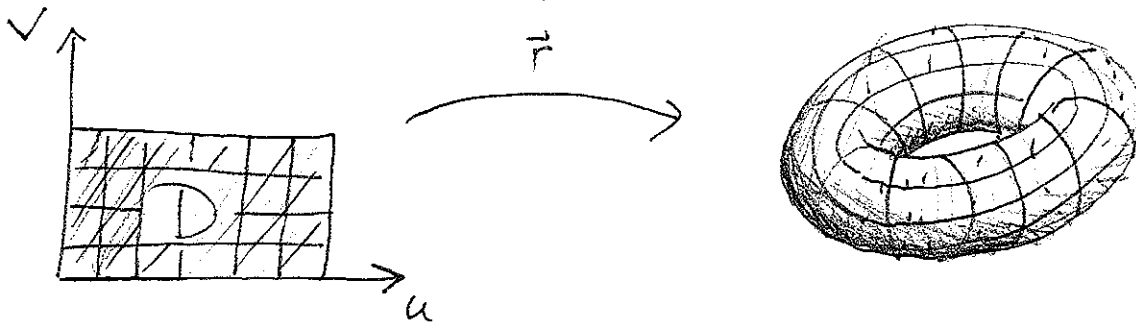


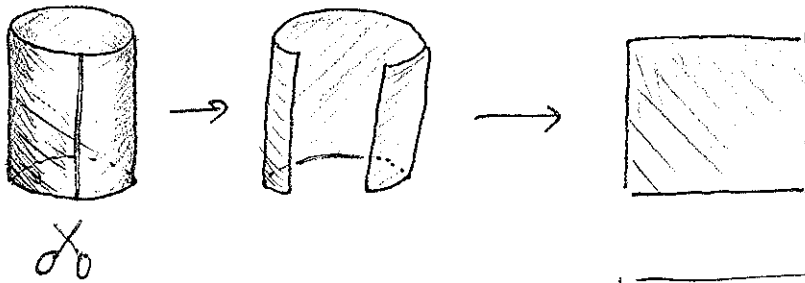
Last time: Parameterizing surfaces



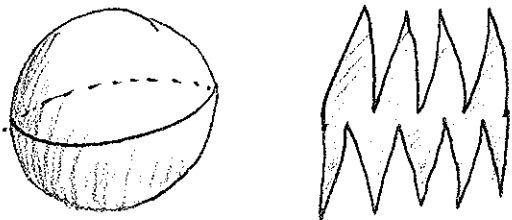
$$\vec{r}: (D \text{ in } \mathbb{R}^2) \longrightarrow (S \text{ in } \mathbb{R}^3)$$

Q: How do we compute the area of S ?

Easy:

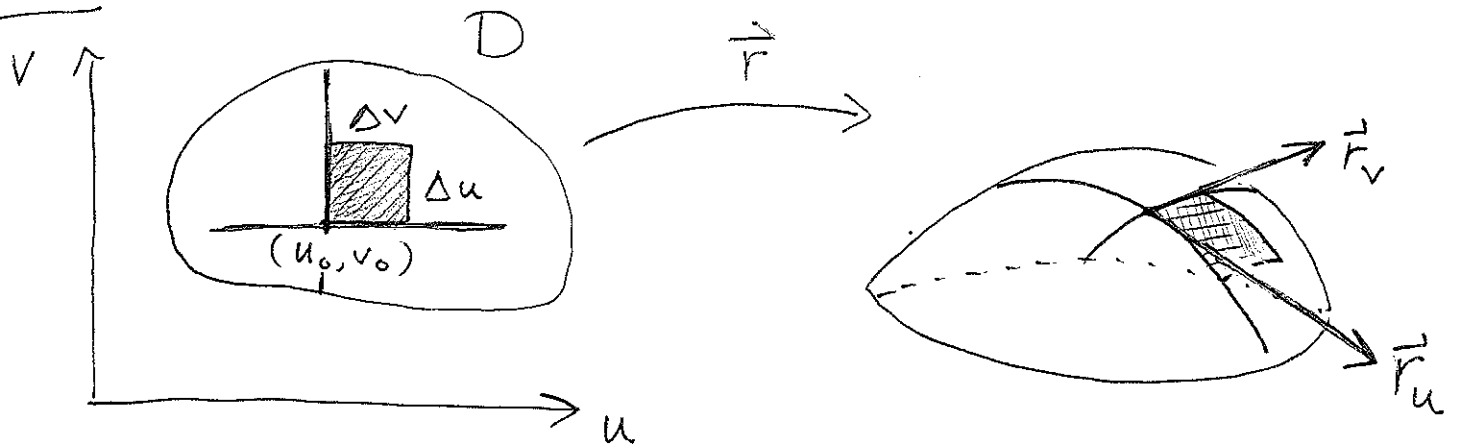


Hard:



Point: Really, the same idea as change of coordinates, basic line ints, etc...

Idea:



We approximate the area of region at right using a parallelogram:

So the area is

$$\approx |(\Delta u \vec{r}_u) \times (\Delta v \vec{r}_v)|$$

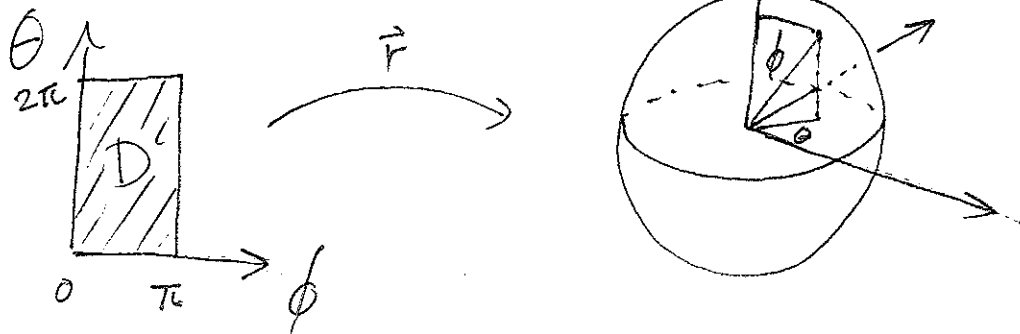
$$= |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v$$

So,

$$\text{Area}(S) \approx \sum_{\text{small nests}} |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v.$$

Let $\Delta u, \Delta v \rightarrow 0$, then $\text{Area}(S) = \iint_D |\vec{r}_u \times \vec{r}_v| du dv$

Ex: Unit sphere



$$\vec{r}(\phi, \theta) = (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi)$$

$$\vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\phi \cos\theta & \cos\phi \sin\theta & -\sin\phi \\ -\sin\phi \sin\theta & \sin\phi \cos\theta & 0 \end{vmatrix}$$

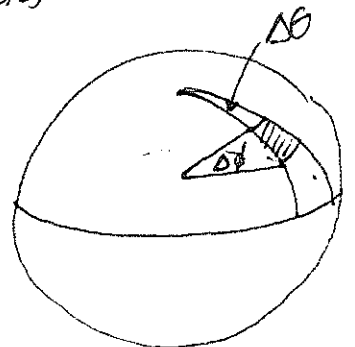
$$= \left(+\sin^2\phi \cos\theta, -\sin^2\phi \sin\theta, \sin\phi \cos\phi (\cos^2\theta + \sin^2\theta) \right)$$

$$|\vec{r}_\phi \times \vec{r}_\theta| = \sqrt{\sin^4\phi + \sin^2\phi \cos^2\phi} = \sqrt{\sin^2\phi} = \sin\phi$$

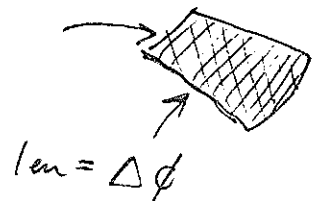
↑ as $0 \leq \phi \leq \pi$.

Geometrically, we saw this $\sin\phi$ before Aside
when we were discussing spherical coordinates

$$\begin{aligned} \text{Area}(\text{Sphere}) &= \iint_D \sin\phi \, d\phi \, d\theta \\ &= \int_0^{2\pi} \int_0^\pi \sin\phi \, d\phi \, d\theta \\ &= \int_0^{2\pi} \left. -\cos\phi \right|_{\phi=0}^{\phi=\pi} d\theta \\ &\quad \rightarrow = -(-1) - (-1) \\ &\quad = 2 \\ &= \int_0^{2\pi} 2 \, d\theta = \boxed{4\pi} \end{aligned}$$



$$\text{len} = \sin\phi \, \Delta\theta$$




$$\text{Area} \approx \sin\phi \, \Delta\theta \, \Delta\phi$$

Note: Area $\left(\begin{array}{c} 1 \\ \text{cylinder} \end{array} \right) = 2 \text{Area}(\text{circle}) + \text{Area}(\text{cylinder})$
 $= 2\pi + 4\pi = 6\pi.$

Hence

$$\frac{\text{Area}(\text{cylinder})}{\text{Area}(\text{circle})} = \frac{3}{2} = \frac{\text{Vol}(\text{cylinder})}{\text{Vol}(\text{circle})}$$

Isn't typical, e.g. compare a cube  with a sphere.

With curves, finding



$$\text{length} = \int_C ds = \int_a^b |\vec{r}'(t)| dt \quad \text{where } \vec{r}: [a, b] \rightarrow C.$$

lead to integrating $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ along C via:

$$\int_C f ds = \int_a^b f(\vec{r}(t)) |\vec{r}'(t)| dt$$

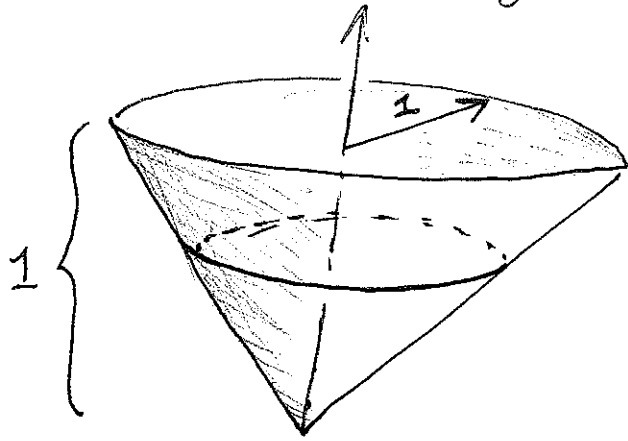
Similarly, if $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ and S is a surface,

$$\iint_S f dA = \iint_D f(\vec{r}(u, v)) |\vec{r}_u \times \vec{r}_v| du dv$$

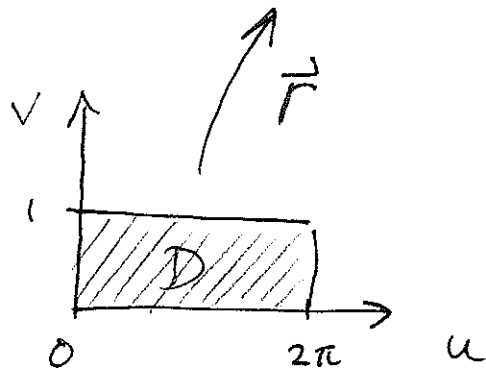
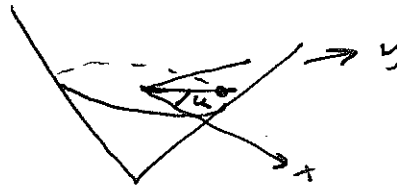
where $\vec{r}: D \rightarrow S$ is a parameterization.



Ex: Find the average of $f(x, y, z) = xy + z$
on the cone S given by $x^2 + y^2 = z^2$ with $0 \leq z \leq 1$.



Parameters: height v angle u



$$\vec{r}(u, v) = (v \cos u, v \sin u, v)$$

Area: $\vec{r}_u = (-v \sin u, v \cos u, 0)$

$$\vec{r}_v = (\cos u, \sin u, 1)$$

$$|\vec{r}_u \times \vec{r}_v| = |(v \cos u, v \sin u, -v)| = \sqrt{2} v$$

So

$$\begin{aligned} \text{Area} &= \iint_S 1 \, dA = \iint_D \sqrt{2} v \, du \, dv = \int_0^1 \int_0^{2\pi} \sqrt{2} v \, du \, dv \\ &= \int_0^1 2\sqrt{2} \pi v \, dv = \sqrt{2} \pi \end{aligned}$$

$$\text{Average} = \frac{1}{\text{Area}} \iint_S xy + z \, dA$$

$$= \frac{1}{\sqrt{2}\pi} \int_0^1 \int_0^{2\pi} (v^2 \sin u \cos u + v) (\sqrt{2} v) \, du \, dv$$

$$= \frac{1}{\pi} \int_0^1 \left(\int_0^{2\pi} v^3 \sin u \cos u + v^2 \, du \right) dv$$

$$= \frac{1}{\pi} \int_0^1 \left(\frac{1}{2} v^3 \sin^2 u + v^2 u \Big|_{u=0}^{u=2\pi} \right) dv$$

$$= \frac{1}{\pi} \int_0^1 2\pi v^2 \, dv = 2 \int_0^1 v^2 \, dv = 2 \frac{v^3}{3} \Big|_{v=0}^1 = \frac{2}{3}.$$