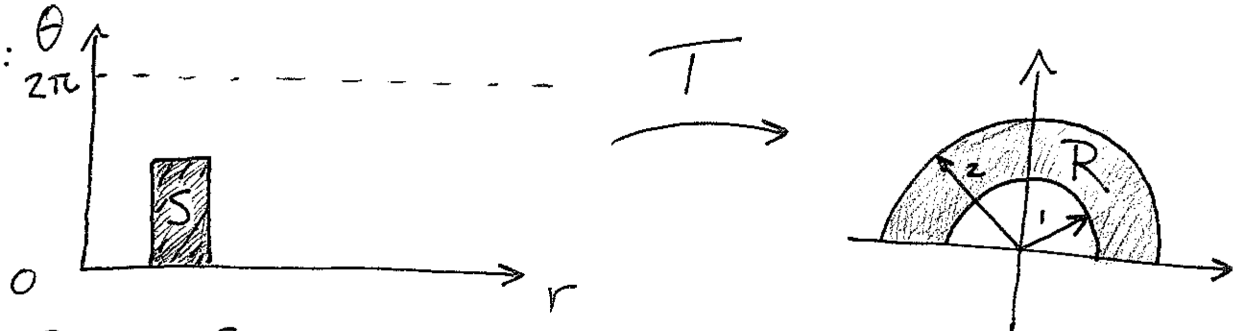
Lecture 30: Changing coordinates I (Section 15.9)

Previously:

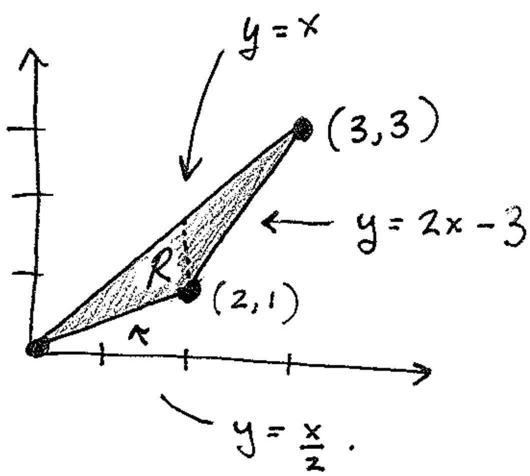


$$T:\mathbb{R}^2 \to \mathbb{R}^2 \quad T(r,\theta) = (r\cos\theta,\ r\sin\theta)$$

$$\iint_R f(x,y)\,dA = \iint_S f(T(r,\theta))\ r\,dr\,d\theta$$

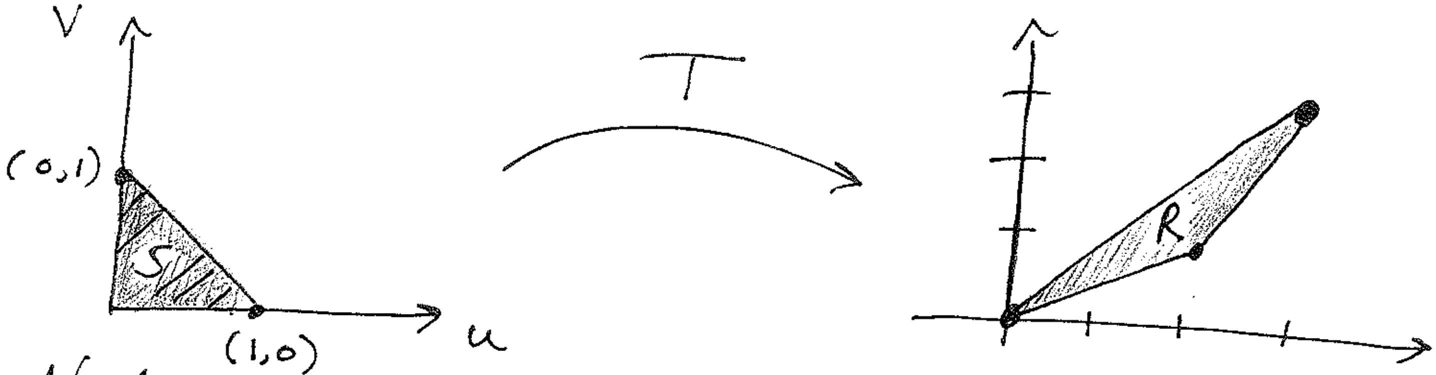
[Next two lectures: general change of coordinates...]

[Suppose we want to integrate many functions over the region shown at right. For each, would need two integrals to do so:]



$$\iint_R f(x,y)\,dA = \int_0^2\int_{\frac{x}{2}}^{x} f(x,y)\,dy\,dx + \int_2^3\int_{2x-3}^{x} f(x,y)\,dy\,dx.$$

[Goal: Do a change of coordinates so that can use just one integral. Sing praises thereof.]



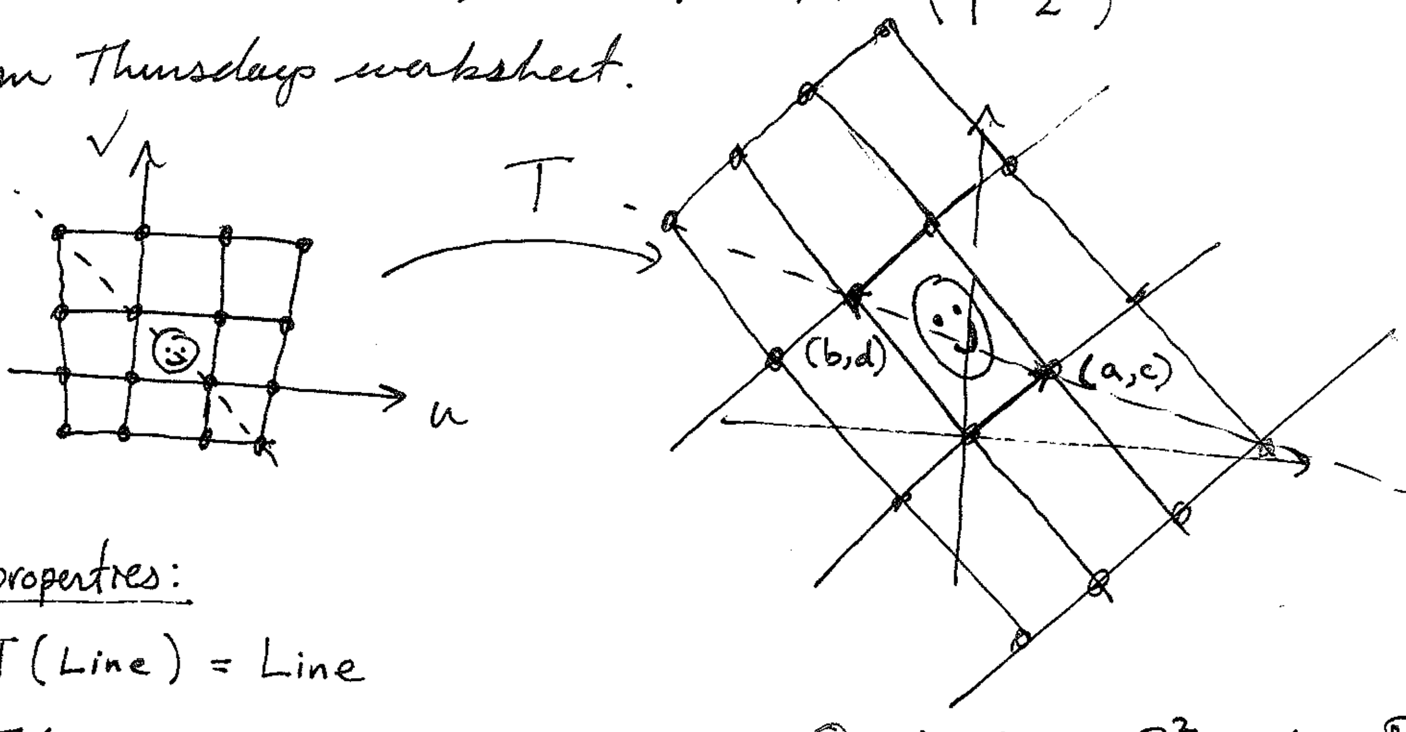
Need $T: \mathbb{R}^2 \to \mathbb{R}^2$ so that $T(S) = R$.

Simplist kind of T: Linear Transformations.

$$T_A(u,v) = (au+bv,\ cu+dv) \quad \text{for some } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

<u>Ex</u>: $T(u,v) = (u-2v,\ u+2v)$ $\quad A = \begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix}$

from Thursdays worksheet.



<u>Key properties</u>:

(a) $T(\text{Line}) = \text{Line}$

(b) $T(0,0) = (0,0)$

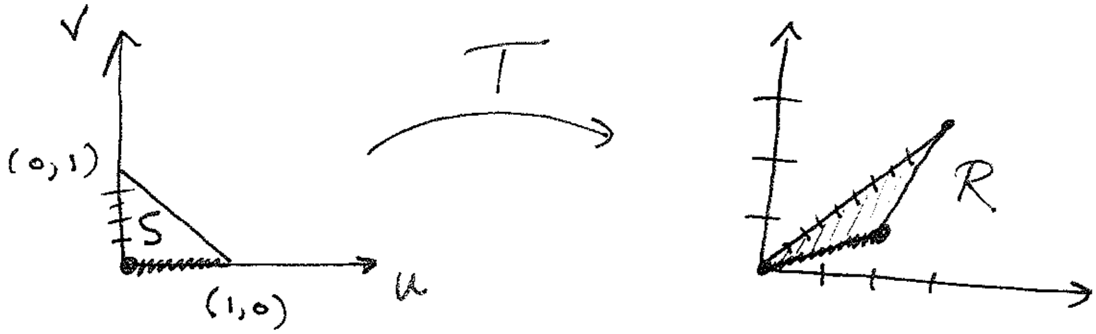
(c) T det. by $T(1,0) = (a,c)$

and $T(0,1) = (b,d)$

(d) $\vec{w}_1, \vec{w}_2$ in $\mathbb{R}^2$, s,t in $\mathbb{R}$

$$T(s\vec{w}_1 + t\vec{w}_2) = sT(\vec{w}_1) + tT(\vec{w}_2)$$

If we want $T(S) = R$

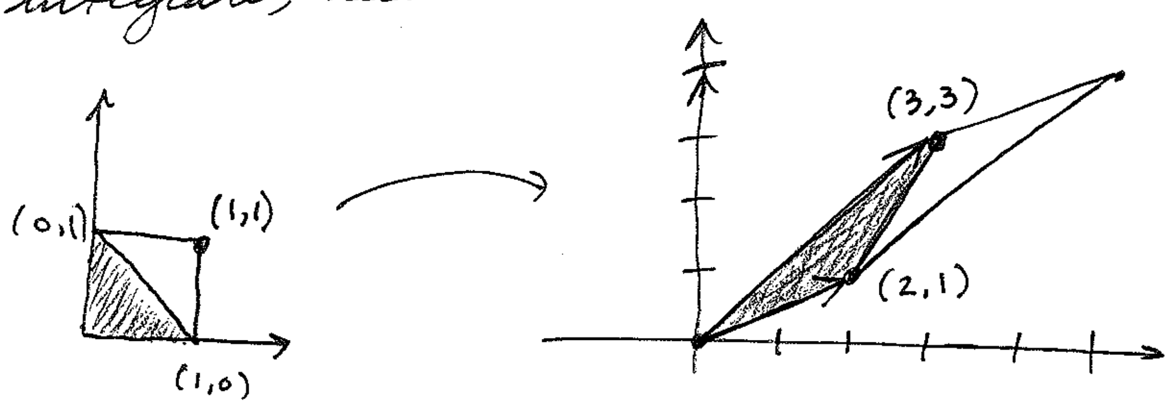


then $T(1,0) = (2,1)$ and $T(0,1) = (3,3)$.

Hence $A = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$ and $T(u,v) = (2u+3v,\ u+3v)$

if T is linear, and that will work since such send lines to lines.

To integrate, need to understand how T distorts area.

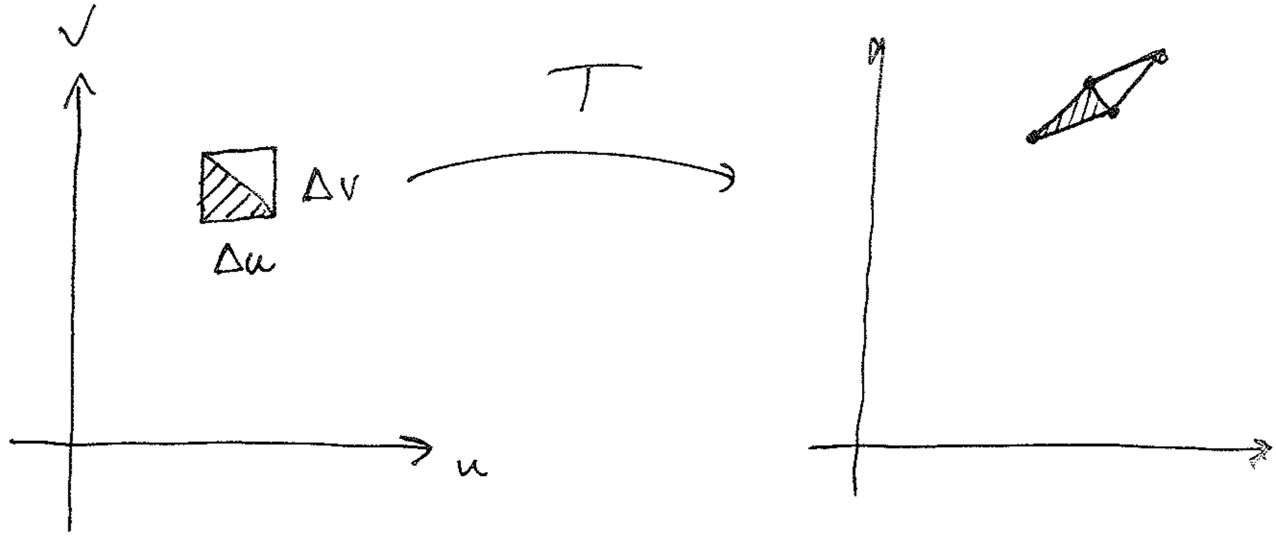


$$\text{Area}(T(\square)) = \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} = 3 \qquad T(1,1) = (5,4)$$

Now a linear transform distorts area uniformly

(think about problem 1 on the worksheet)

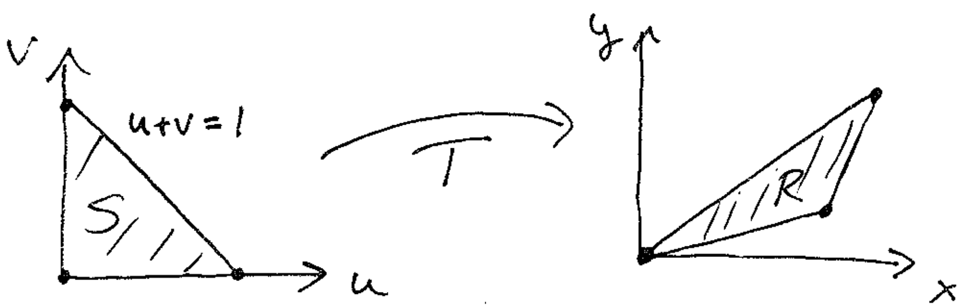
Hence



$$\text{Area}(\square) = \Delta u \, \Delta v \longrightarrow \text{Area}(\text{▱}) = 3 \, \Delta u \, \Delta v$$

So: $dA = 3 \, du \, dv$

Ex: $$\iint_R x - y \, dA = \iint_S (2u+3v) - (u+3v) \; 3 \, du \, dv$$



$$T(u,v) = (2u+3v, u+3v) = (x,y)$$

$$= \int_0^1 \int_0^{1-u} u \;\; 3 \, dv \, du = \int_0^1 3u(1-u) \, du$$

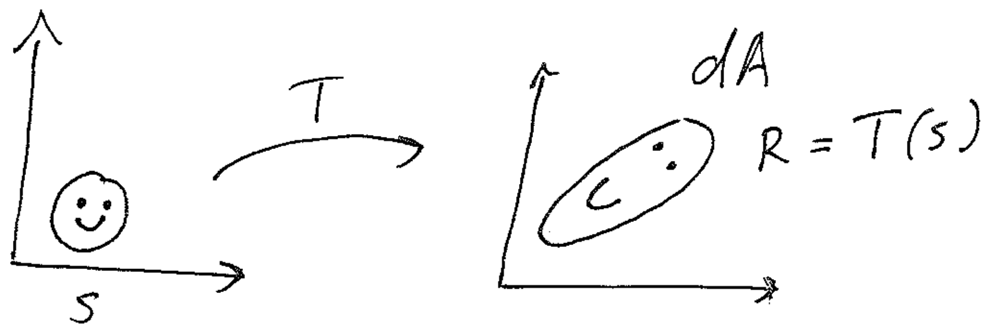
$$= \int_0^1 3u - 3u^2 \, du = \frac{3}{2}u^2 - u^3 \Big|_{u=0}^{u=1} = \frac{1}{2}.$$

Fun check: Do as shown on pg. 89.

In general, if $T: \mathbb{R}^2 \to \mathbb{R}^2$ is linear,

with $T(u,v) = (au+bv, cu+dv)$ for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

then



and $dA = \begin{vmatrix} a & b \\ c & d \end{vmatrix} du\,dv$.

Thus

$$\iint_R f(x,y)\,dA = \iint_S f(T(u,v)) \begin{vmatrix} a & b \\ c & d \end{vmatrix} du\,dv$$

Linear Approx: Recall if $g: \mathbb{R}^2 \to \mathbb{R}$ is diff at (u,v) then

$$g(u+\Delta u, v+\Delta v) = g(u,v) + g_u(u,v)\Delta u + g_v(u,v)\Delta v + \underbrace{E(\Delta u, \Delta v)}_{\text{small}}.$$

Now consider $T: \mathbb{R}^2 \to \mathbb{R}^2$

$$T(u,v) = (g(u,v), h(u,v)).$$

Then

$$T(u+\Delta u, v+\Delta v) = T(u,v) + J_{(u,v)}(\Delta u, \Delta v) + \text{Error}$$

where

$J_{(u,v)}$ is the linear transformation with matrix $\begin{pmatrix} g_u(u,v) & g_v(u,v) \\ h_u(u,v) & h_v(u,v) \end{pmatrix}$

Reason

$$T(u+\Delta u, v+\Delta v)$$

$$\approx \big(g(u,v) + g_u(u,v)\Delta u + g_v(u,v)\Delta v,\ h(u,v) + h_u(u,v)\Delta u + h_v(u,v)\Delta v \big)$$

Thus locally T looks like a linear transformation...

— to be continued —