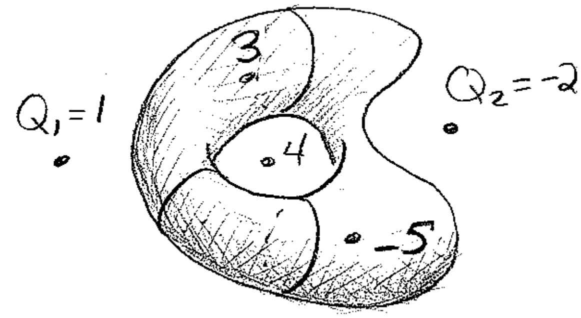
Lecture 43: Maxwell's Equations

Last time: Charges Q_i at pos. $\vec{p}_i$

$$\vec{E}(\vec{r}) = \sum \frac{Q_i}{4\pi\varepsilon_0} \frac{1}{|\vec{r}-\vec{p}_i|^3}(\vec{r}-\vec{p}_i)$$

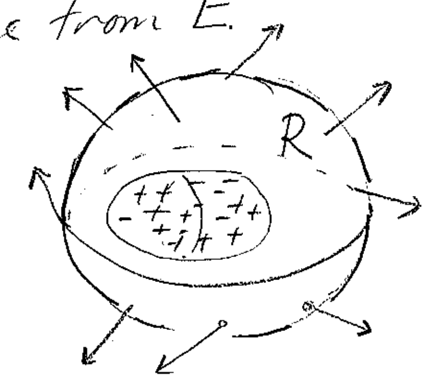


$$\text{Flux} = -\frac{2}{\varepsilon_0}$$

Gauss's Law: R region in $\mathbb{R}^3$

$$\iint_{\partial R} (\vec{E}\cdot\vec{n})\, dA = \frac{1}{\varepsilon_0}\left(\begin{array}{c}\text{Total charge}\\ \text{in } R\end{array}\right)$$

[In HW, used this to compute total charge from $\vec{E}$.



This is not impractical...

When there are many (e.g. 10^{20}) charged particles, can't use the formula for $\vec{E}$ directly, and anyway it doesn't make sense to focus on them individually, just like we don't count molecules when measuring mass.]

Mass: $\rho(x,y,z)$ mass density, units $= \frac{g}{m^3}$

$$\text{Total mass of } R = \iiint_R \rho\, dV$$

R

Charge: $\rho(x,y,z)$ charge density, units $= \frac{\text{coulomb}}{m^3}$

$$\text{Total Charge in } R = \iiint_R \rho \, dV$$

<u>Q</u>: How does ρ determine $\vec{E}$?

Gauss's Law should still hold, so for <u>any</u> region R

$$\frac{1}{\varepsilon_0}\left(\text{Total Charge in } R\right) = \iint_{\partial R} (\vec{E}\cdot\vec{n})\, dA \underset{\text{Divergence Thm.}}{=} \iiint_R \operatorname{div}\vec{E}\, dV$$

$$\parallel$$

$$\frac{1}{\varepsilon_0}\iiint_R \rho \, dV \qquad \Longrightarrow \text{ Must have } \boxed{\operatorname{div}\vec{E} = \frac{\rho}{\varepsilon_0}}$$

[<u>Q</u>: Does this answer the question? Not completely as many vector field have the <u>same</u>.]

<u>A</u>: $\vec{E}(\vec{r}) = (E_1(\vec{r}),\ E_2(\vec{r}),\ E_3(\vec{r}))$ where if $\vec{r} = (a,b,c)$ then e.g.

$$E_1(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\iiint_R \frac{(a-x)\,\rho(x,y,z)}{|\vec{r}-(x,y,z)|^2}\, dV$$

Maxwell's Equations:

$\vec{E}(x,y,z,t)$ – Electric field $\qquad \rho(x,y,z,t)$ – charge density.

$\vec{B}(x,y,z,t)$ – Magnetic field

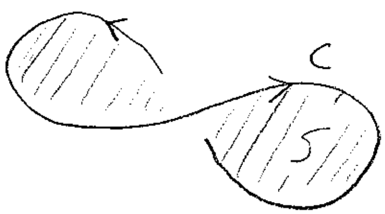
Gauss's Law:

$$\operatorname{div}\vec{E} = \frac{\rho}{\varepsilon_0} \qquad \iint_{\partial R} \vec{E}\cdot\vec{n}\; dA = \iiint_R \rho\, dV$$

Gauss's Law for Magnetic Fields: [no magnetic monopoles]

$$\operatorname{div}\vec{B} = 0 \qquad \iint_{\partial R} (\vec{B}\cdot\vec{n})\, dA = 0$$

Faraday's Law of Induction: A changing magnetic field induces a current in a loop of wire



$$\underbrace{\int_C \vec{E}\cdot d\vec{r}}_{\text{electromotive force aka voltage}} = -\frac{\partial}{\partial t}\iint_S (\vec{B}\cdot\vec{n})\, dA$$

Unit check: $\vec{F} = Q\cdot\vec{E} \Rightarrow \vec{E}$ in $\frac{N}{C} = \frac{V}{m}$, so $\int \vec{E}\cdot \underbrace{d\vec{r}}_{m}$ is in volts

$\vec{B}$ has units $T = \text{Tesla} = \frac{Vs}{m^2}$ $\qquad \vec{n}$ – unitless

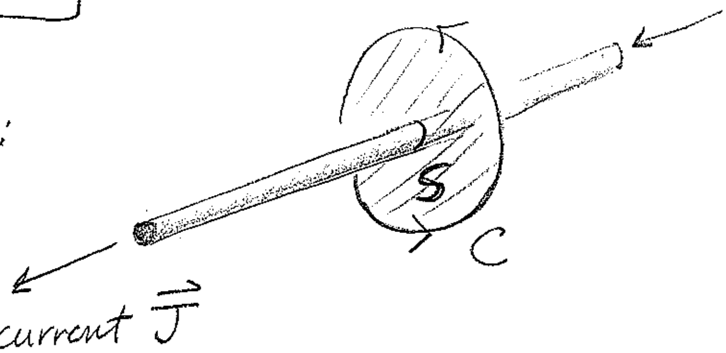
$\iint_S (\vec{B}\cdot\vec{n})\underset{m^2}{dA}$ in Vs so $\frac{\partial}{\partial t}\iint_S (\vec{B}\cdot\vec{n})\, dA$ in V.

$\vec{J}$ = current density in $\frac{\text{Amps}}{m^2}$

Now $\int_C \vec{E}\cdot d\vec{r} = \iint_S (\text{curl } \vec{E})\cdot\vec{n}\, dA$, and so

$$\boxed{\text{curl } \vec{E} = \frac{\partial \vec{B}}{\partial t}}$$

Ampere's circuital law:



$$\int_C \vec{B}\cdot d\vec{r} = \mu_0 \iint_S (\vec{J}\cdot\vec{n})\, dA + \varepsilon_0 \mu_0 \frac{\partial}{\partial t} \iint_S (\vec{E}\cdot\vec{n})\, dA$$

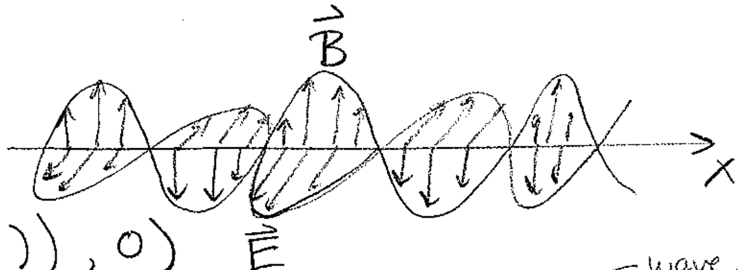
$$\parallel$$

$$\iint_S (\text{curl } \vec{B})\cdot\vec{n}\, dA$$

$$\boxed{\text{curl } \vec{B} = \mu_0 \vec{J} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}}$$

And there was light:

c = speed of light $= \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$



$\vec{E} = (0,\ c\cdot\cos(2\pi(\omega t - x/\lambda)),\ 0)$

$\vec{B} = (0,\ 0,\ \cos(2\pi(\omega t - x/\lambda)))$

$c = \omega\lambda$ ← wave length

↑ freq.

Check: $\text{curl } \vec{E} = \frac{\partial \vec{B}}{\partial t}$ $\quad$ $\text{curl } \vec{B} = \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$

$\text{div } \vec{E} = 0$ $\quad$ $\text{div } \vec{B} = 0$.

If time remains
$$\int_{\partial M} \omega = \int_M d\omega$$

[Implicit point: Hences another application of line and surface integrals.]