

Lecture 42: Electrostatics and Gauss's Law

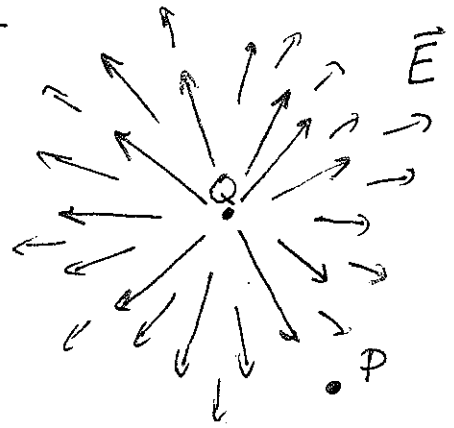
Previously: Divergence Thm: D a region in \mathbb{R}^3 , \vec{F} a vector field

$$\iint_{\partial D} (\vec{F} \cdot \vec{n}) dA = \iiint_D \operatorname{div} \vec{F} dV$$

Electrostatics: Particle of charge Q at \vec{o} .

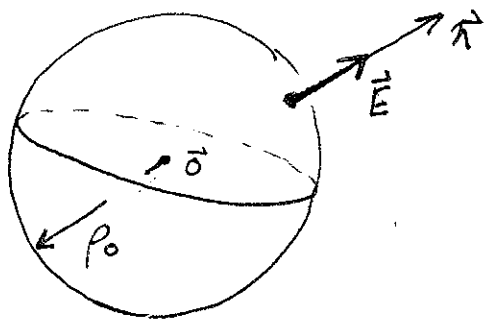
Electric field at $\vec{r} = (x, y, z)$:

$$\vec{E}(\vec{r}) = \frac{Q}{4\pi\epsilon_0} \frac{1}{|\vec{r}|^3} \vec{r} \quad \left[\begin{array}{l} \text{inverse} \\ \text{square law} \end{array} \right]$$



Force experienced by a charge P at position \vec{r} : $\vec{F} = PE(\vec{r})$.

Flux:



$S =$ sphere of radius ρ_0

$$\begin{aligned} \iint_S (\vec{E} \cdot \vec{n}) dA &= \iint_{S_{\rho_0}} |\vec{E}| dA \\ &= \iint_S \frac{Q}{4\pi\epsilon_0 \rho_0^2} dA = \frac{Q}{\epsilon_0} \end{aligned}$$

Note: We calculated $\text{Area}(\odot) = 4\pi$, so
 $\text{Area}(\odot_{\rho_0}) = 4\pi\rho_0^2$.

Let D be the region bounded by S , and let's check this by calculating $\iiint_D \operatorname{div} \vec{F} dV$

Now $\vec{E} = \frac{Q}{4\pi\epsilon_0} \left(\frac{x}{\rho^3}, \frac{y}{\rho^3}, \frac{z}{\rho^3} \right)$ where $\rho = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

and noting that $\frac{\partial \rho}{\partial x} = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} (2x) = \frac{x}{\rho}$

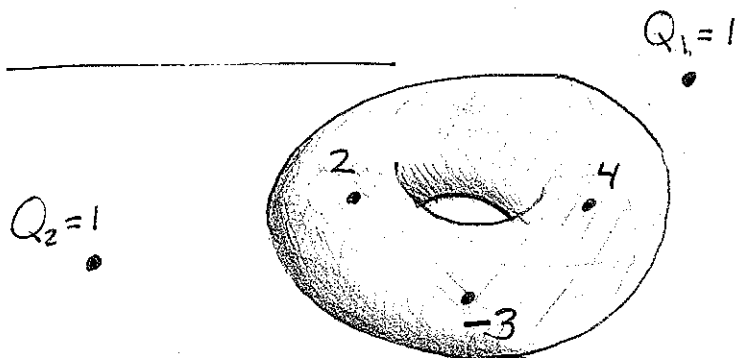
we find

$$\text{div } \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{Q}{4\pi\epsilon_0} \left(\frac{\rho^3 - 3\rho^2 x^2}{\rho^6} + \frac{\rho^3 - 3\rho^2 y^2}{\rho^6} + \frac{\rho^3 - 3\rho^2 z^2}{\rho^6} \right) = 0.$$

So it appears that $\iiint_D \text{div } \vec{E} dV = 0$, violating the Divergence Thm???. What's gone wrong?

A. \vec{E} isn't defined on all of D [since it doesn't make sense at $\vec{0}$.]

Suppose we have charges Q_i at positions \vec{p}_i



$$\vec{E}(\vec{r}) = \sum E_i(\vec{r}) = \sum \frac{Q_i}{4\pi\epsilon_0} \frac{1}{|\vec{r} - \vec{p}_i|^3} (\vec{r} - \vec{p}_i)$$

Gauss's Law: Let D be a region in \mathbb{R}^3 . Then

$$\iint_{\partial D} (\vec{E} \cdot \vec{n}) dA = \frac{1}{\epsilon_0} \left(\sum_{\substack{i \text{ where } \vec{p}_i \\ \text{is in } D}} Q_i \right)$$

Ex: In the above pic, $\iint_{\partial D} (\vec{F} \cdot \vec{n}) dA = \frac{1}{\epsilon_0} (2+4-3) = \frac{3}{\epsilon_0}$ 123

Reason [for Gauss's Law:] First notice

$$\iint_{\partial D} (\vec{E} \cdot \vec{n}) dA = \iint_{\partial D} (\sum E_i) \cdot \vec{n} dA = \sum \iint_{\partial D} (E_i \cdot \vec{n}) dA$$

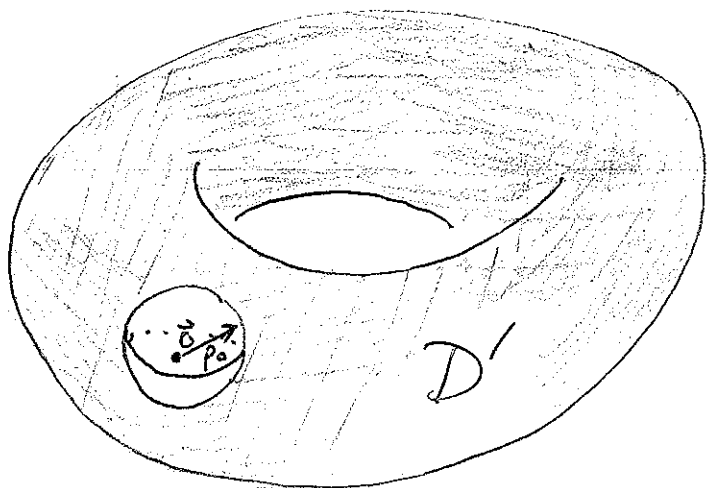
So assume we have a single charge Q at \vec{o} .

(after changing coordinates), and hence $\vec{E} = \frac{Q}{4\pi\epsilon_0} \frac{1}{\rho^3} \vec{r}$

Key: $\iint_{\partial D} (\vec{E} \cdot \vec{n}) dA = \begin{cases} 0 & \text{if } \vec{o} \text{ is not in } D. \\ Q/\epsilon_0 & \text{if } \vec{o} \text{ is in } D. \end{cases}$

Point: If \vec{o} is not in D , then $\text{div } \vec{E} = 0$ on all of D , and hence $\iint_{\partial D} (\vec{E} \cdot \vec{n}) dA = \iiint_D \text{div } \vec{E} dV = 0$.

If \vec{o} is in D , consider $D' = D$ with ball of radius ρ_0 about \vec{o} removed.



Then

$$0 = \iiint_{D'} \text{div } \vec{E} dV =$$

$$\iint_{\partial D'} (\vec{E} \cdot \vec{n}) dA =$$

$$\iint_{\partial D} \vec{E} \cdot \vec{n} \, dA + \iint_{S_{p_0}} \vec{E} \cdot \vec{n} \, dA$$

↑
outward normal
↙
inward normal

So

$$\iint_{\partial D} \vec{E} \cdot \vec{n} \, dA = \iint_{S_{p_0}} \vec{E} \cdot \vec{n} \, dA = \frac{Q}{\epsilon_0}$$

↙
outward normal.

While charge is quantized, when there are many charges, look at the charge density: $\rho(x, y, z)$ (units: $\frac{\text{coulomb}}{\text{m}^3}$). By Gauss's Law

$$\iint_{\partial D} (\vec{E} \cdot \vec{n}) \, dA = \frac{1}{\epsilon_0} \iiint_D \rho \, dV$$

||

$$\iiint_D \text{div } \vec{E} \, dV$$

Since this is true for all D , must have

$$\text{div } \vec{E} = \frac{1}{\epsilon_0} \rho.$$

