Lecture 41: Conservative Vector Fields in $\mathbb{R}^3$

Previously: A vector field $\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$ is conservative if $\vec{F} = \nabla f$ for some $f: \mathbb{R}^n \to \mathbb{R}$.

Ex: $\vec{F} = (y, x)$ as if $f = xy$ then $\nabla f = (y, x)$

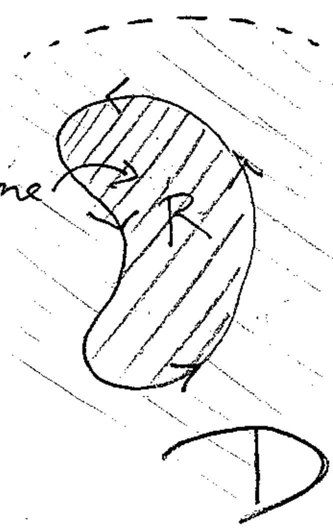
Non Ex: $\vec{F} = (-y, x)$ since $\frac{\partial x}{\partial x} = 1 \neq -1 = \frac{\partial(-y)}{\partial y}$.

Thm A: A vector field $\vec{F}$ on D in $\mathbb{R}^n$ is conservative if and only if $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed curve C in D.

Thm B: If D is in $\mathbb{R}^2$ is simply connected, then $\vec{F} = (P, Q)$ is conservative if and only if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$.

Missing Link: If $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ why must $\vec{F}$ be conservative?

Reason: Green's Thm If C in D is closed then as D has no holes it is the boundry of some



Thus:

$$\int_C \vec{F} \cdot d\vec{r} = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA$$

$$= \iint_R 0 \, dA = 0.$$

Thus Thm A applies.

[Want a Thm B for $\mathbb{R}^3$...]

Suppose $\vec{F} = \nabla f = (f_x, f_y, f_z)$

Then

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \overbrace{\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)}^{=0}\vec{i} - \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right)\vec{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\vec{k}$$

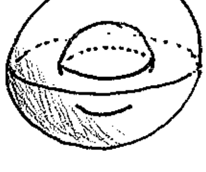
$$= \vec{0}$$

<u>Ex</u>: $\vec{F} = (y, z, x)$ is not conservative, since $\operatorname{curl}\vec{F} = (1, -1, -1)$.

[Is $\operatorname{curl}\vec{F} = \vec{0}$ enough.]

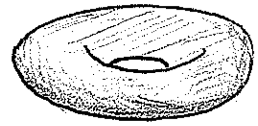
<u>Thm</u>: A vector field $\vec{F}$ on all of $\mathbb{R}^3$ is conservative if and only if $\operatorname{curl}\vec{F} = \vec{0}$ everywhere.

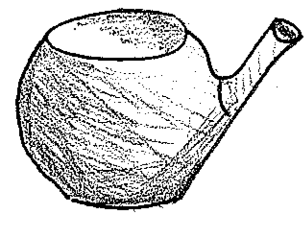
More generally true when D is simply connected.



$1 \le x^2 + y^2 + z^2 \le 2$

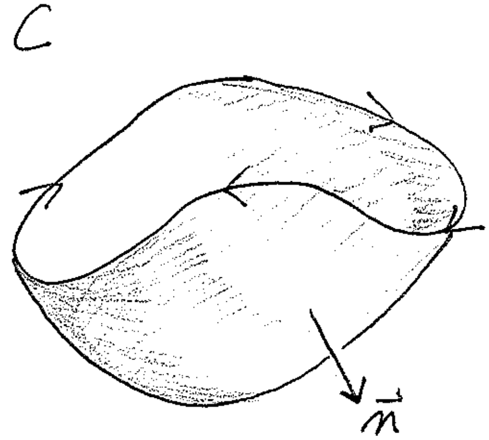
Simply Conn.





Not simply con.

Reason: Suppose $\operatorname{curl}\vec{F} = \vec{0}$ and C is a closed curve in $\mathbb{R}^3$. [Need $\int_C \vec{F}\cdot d\vec{r} = 0$ so Thm A applies.] Suppose S is an orientable surface with $\partial S = C$.



By Stokes:

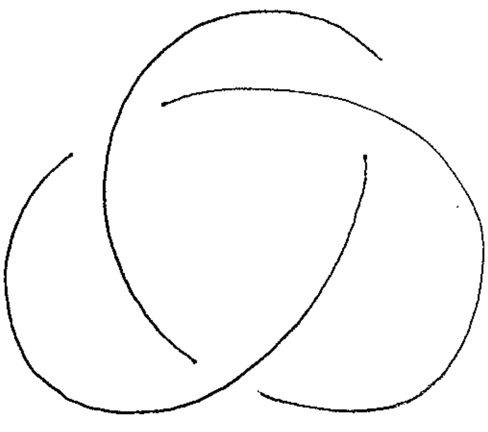
$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\operatorname{curl}\vec{F})\cdot\vec{n}\, dA$$

$$= \iint_S (\vec{0}\cdot\vec{n})\, dA = 0$$

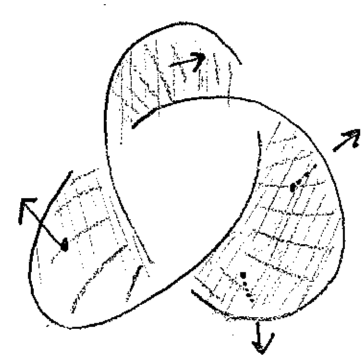
Q: Is every C the boundary of some S?

If yes, $\int_C \vec{F}\cdot d\vec{r} = 0$ for all C, and so Thm A gives that $\vec{F}$ is conservative.

Ex:



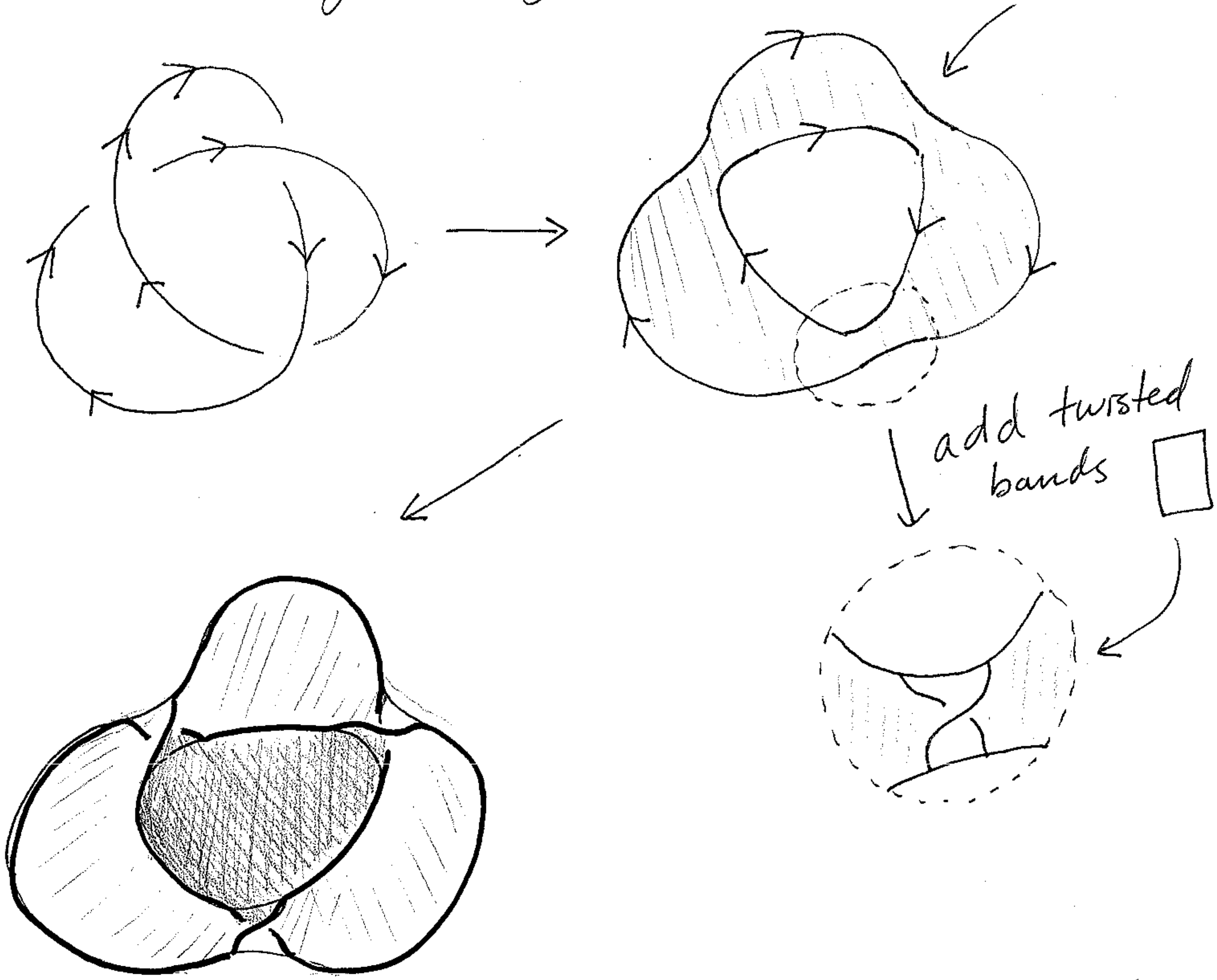
Try 1:



Problem: not orientable

In fact, there is always such a surface
[can be found algorithmically.]

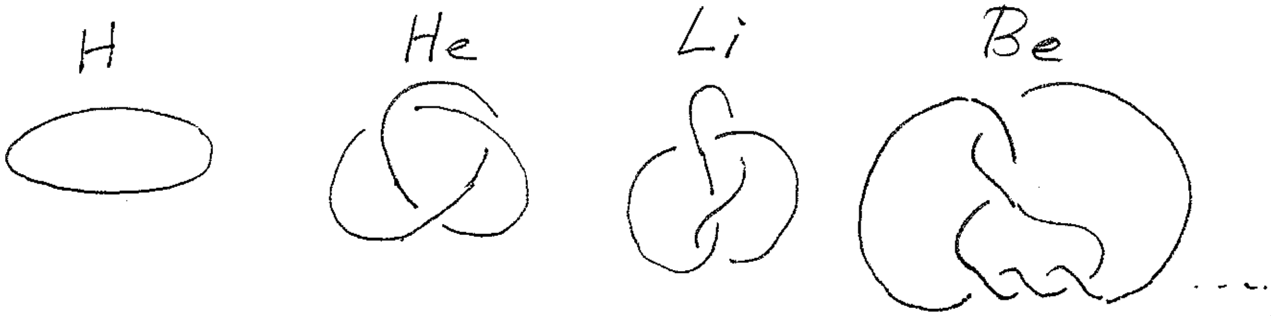
Two discs.



which is actually orientable (normals point at you for both discs).

A closed curve (w/o self intersections) is called a <u>knot</u>. Part of a branch of mathematics called <u>topology</u>.

<u>History</u>: Lord Kelvin: atoms as knots in the "ether"
Tait: made a table of simple knots (1870s)



Actually, theres no ether. (Michleson - Morley 1880s)
Einstein 1905.

Mathematicians thought about knots anyway for 100 years. Now used in biology studying action of enzymes on DNA....

Public key cryptography...

Least area surfaces...