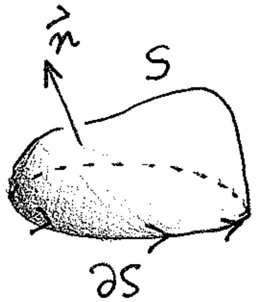
Lecture 40: Applications of Stokes' Thm

- http://dunfield.info/241/final.html

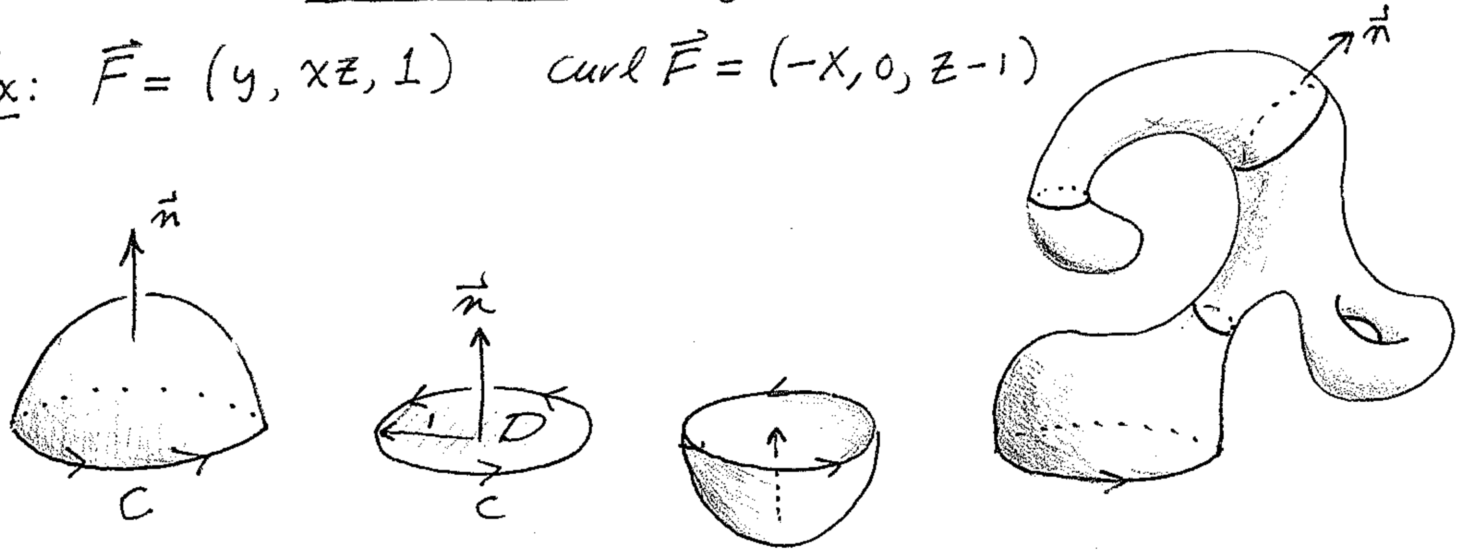
Last time: Stokes' Thm: S a surface in $\mathbb{R}^3$

$\vec F : \mathbb{R}^3 \to \mathbb{R}^3$ a vector field. Then



$$\int_{\partial S} \vec F \cdot dr = \iint_S (\mathrm{curl}\, \vec F) \cdot \vec n \; dA$$

Ex: $\vec F = (y, xz, 1) \qquad \mathrm{curl}\, \vec F = (-x, 0, z-1)$



$$\iint (\mathrm{curl}\, \vec F) \cdot \vec n \; dA = -\pi \text{ for \underline{all} of these!}$$

$$= \int_C \vec F \cdot dr$$

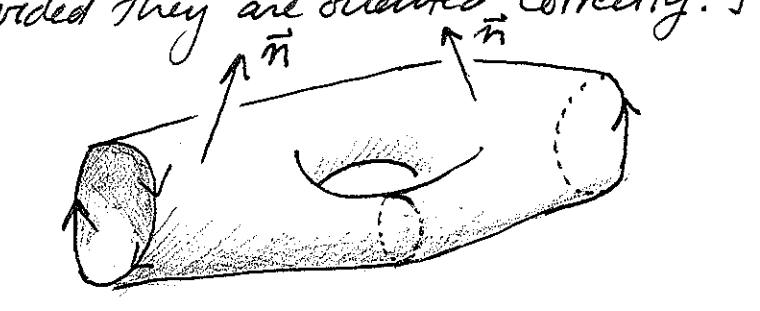
Takes some getting used to... really just Green in disguise...

Easy to check for D:

$$\iint_D (\mathrm{curl}\, \vec F) \cdot \vec n \; dA = \iint_D (-x, 0, -1) \cdot (0, 0, 1) \; dA =$$

$$= \iint_D -1 \; dA = -\mathrm{Area}(D) = -\pi \;\checkmark$$

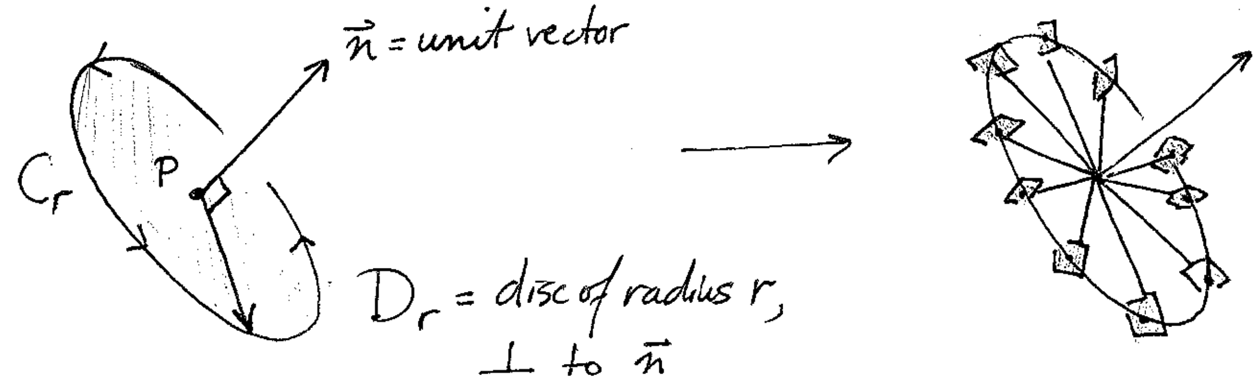
Note: Stokes' Thm also works when S has several boundary components [provided they are oriented correctly.]



Understanding Curl:

$\vec{F} =$ velocity of fluid flow

Consider a small paddle wheel:

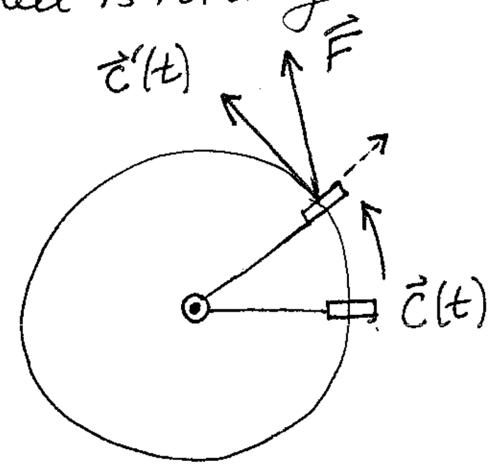


Key: Wheel rotates at

$$\omega = \frac{1}{2\pi r^2} \int_{C_r} \vec{F} \cdot d\vec{r}$$

Reason: Suppose $\vec{n} = (0,0,1)$ and the wheel is rotating at a constant rate ω:

$$\vec{c}(t) = (r\cos \omega t, r \sin \omega t, 0)$$



If the tangential component of $\vec{F}$ has the same length everywhere, then

$$\vec{c}'(t) = \operatorname{proj}_{\vec{c}'(t)} \vec{F} = \frac{\vec{F}\cdot\vec{c}'(t)}{|\vec{c}'(t)|^2}\,\vec{c}'(t)$$

i.e. $\vec{F}\cdot\vec{c}'(t) = |\vec{c}'(t)|^2$. In general the <u>averages</u> of these should be the same, so

$$\int_0^{2\pi/\omega} \vec{F}\cdot\vec{c}'(t)\,dt = \int_0^{2\pi/\omega} |\vec{c}'(t)|^2\,dt = \int_0^{2\pi/\omega} r^2\omega^2\,dt$$

$$= 2\pi r^2 \omega$$

$$\int_{C_r} \vec{F}\cdot d\vec{r} = \int_0^{2\pi/\omega} \vec{F}\cdot\vec{c}'(t)\,dt$$

$$\boxed{\text{Thus} \quad \omega = \frac{1}{2\pi r^2}\int_{C_r} \vec{F}\cdot d\vec{r}}$$

By Stokes:

$$\omega = \frac{1}{2\pi r^2}\iint_{D_r} (\operatorname{curl}\vec{F})\cdot\vec{n}\,dA$$

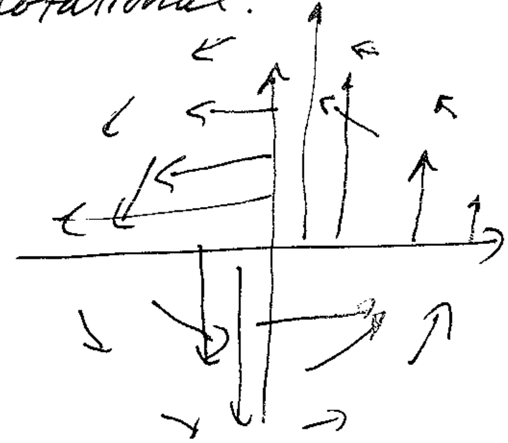
$$= \frac{1}{2}\left(\frac{1}{\text{Area}(D_r)}\iint_{D_r} (\operatorname{curl}\vec{F})\cdot n\,dA\right)$$

Taking $r \to 0$, get

$$\omega = \frac{1}{2}(\operatorname{curl}\vec{F}(p))\cdot\vec{n}$$

Thus the rate of rotation is largest in the direction of curl $\vec{F}$ and then $\omega = \frac{1}{2}|\text{curl } \vec{F}(p)|$.

<u>Note</u>: Vector fields with curl $\vec{F} = \vec{0}$ are called <u>irrotational</u>. Oddly, they can still rotate; expimentally a draining tub is irrotational.



<u>Ex</u>: $\vec{F}(x,y,z) = \frac{1}{x^2+y^2}(-y, x, 0)$

<u>Check</u>: curl $\vec{F} = \vec{0}$ except at $\vec{0}$ where it doesn't make sense.

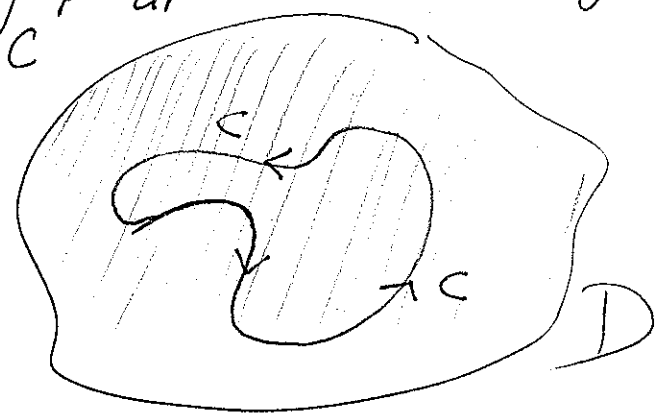
<u>Conservative Vector Fields</u>: $\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$ is conservative if $\vec{F} = \nabla f$ for some $f: \mathbb{R}^n \to \mathbb{R}$

<u>Ex</u>: $\vec{F} = (x, y)$ conservative since $= \nabla\left(\frac{1}{2}(x^2+y^2)\right)$

$\vec{F} = (-y, x)$ not since $\frac{\partial Q}{\partial x} = 1 \neq -1 = \frac{\partial P}{\partial y}$

P Q

<u>Thm A</u>: $\vec{F}$ on a connected set D in $\mathbb{R}^n$ is conservative if and only if $\int_C \vec{F}\cdot d\vec{r} = 0$ for every closed curve C.



<u>Thm B</u>: If D in $\mathbb{R}^2$ is simply connected (no holes), then $\vec{F} = (P, Q)$ is conservative if and only if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$.

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$$

<u>Missing Link</u>: If $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$, then $\int_C \vec{F}\cdot d\vec{r} = 0$ for each closed curve C.

<u>Reason</u>: As D has no holes, the curve C is the boundary of some region R.

Then

$$\int_C \vec{F}\cdot dr = \iint_R \overbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}^{=0!}\, dA$$

$$= \iint_R 0 \, dA = 0.$$

Next time: What is Theorem B for $\mathbb{R}^3$?

A start: Suppose $\vec{F} = \nabla f = (f_x, f_y, f_z)$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \Big(\underbrace{\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}}_{=0},\ 0,\ 0 \Big)$$

$$= \vec{0}.$$

Q: Is this enough?