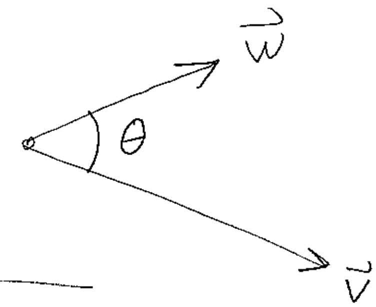
Lecture 4: Cross Product (Section 12.4)

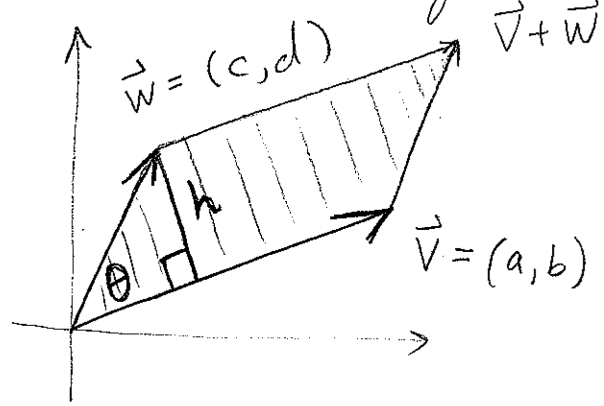
Last time: $\vec{v} = (v_1, v_2, v_3)$, $\vec{w} = (w_1, w_2, w_3)$

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$$
$$= |\vec{v}||\vec{w}| \cos\theta$$



Determinant: $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Meaning: Area of

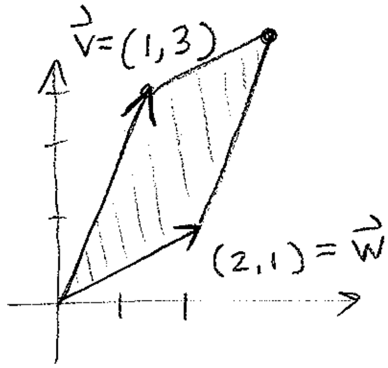


$$A = (\text{base})(\text{height})$$
$$= |\vec{v}||\vec{w}| \sin\theta$$

Reason:

$$A^2 = |\vec{v}|^2 |\vec{w}|^2 \sin^2\theta = |\vec{v}|^2 |\vec{w}|^2 (1 - \cos^2\theta)$$
$$= |\vec{v}|^2 |\vec{w}|^2 - (\vec{v} \cdot \vec{w})^2$$
$$= (a^2 + b^2)(c^2 + d^2) - (ac + bd)^2$$
$$= a^2 d^2 + b^2 c^2 - 2abcd = (ad - bc)^2 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}^2$$

Ex:



$$\text{Area} = \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 1 - 2\cdot 3 = -5$$

negative area ??

Switch $\vec{v}$ and $\vec{w}$: $\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 2\cdot 3 - 1 = 5$

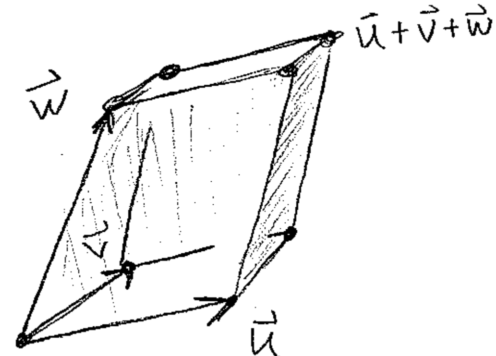
In general

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{cases} \text{area} & \text{if } \vec{w} \text{ is anticlockwise from } \vec{v} \\ -\text{area} & \text{if } \vec{w} \text{ is clockwise from } \vec{v}. \end{cases}$$

[First example of the right-hand rule.]

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

= Volume in $\mathbb{R}^3$ of this parallelepiped.



Cross product: $\vec{v} = (v_1, v_2, v_3) = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$

$\vec{w} = (w_1, w_2, w_3)$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}\vec{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}\vec{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}\vec{k}$$

Ex: $\vec{v} = (0,1,3)$ $\vec{w} = (2,1,1)$

$$\vec{v}\times\vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 3 \\ 2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix}\vec{i} - \begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix}\vec{j} + \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix}\vec{k}$$

$$= -2\vec{i} + 6\vec{j} - 2\vec{k} = (-2, 6, -2)$$

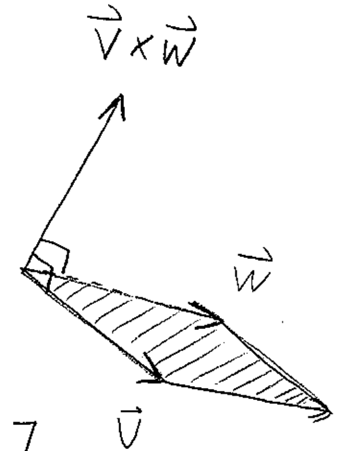
Usefulness: Maxwell's Equations, torque, geom. prob.

Properties:

1) $\vec{v}\times\vec{w}$ is orthogonal to $\vec{v}$ and $\vec{w}$

2) $|\vec{v}\times\vec{w}| = |\vec{v}||\vec{w}|\sin\theta$

= area of pallelogram



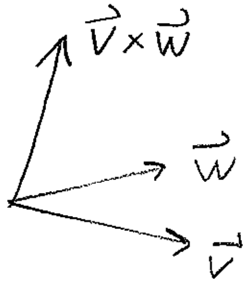
3) $\vec{v}\times\vec{w} = -\vec{w}\times\vec{v}$ [Not commutative!]

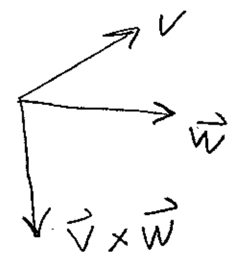
4) $\vec{v}\times\vec{v} = \vec{0}$

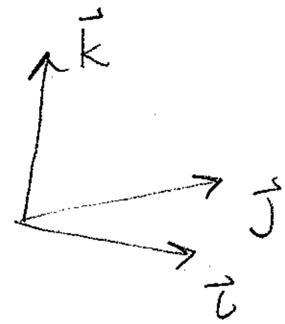
5) $\vec{u}\times(\vec{v}+\vec{w}) = \vec{u}\times\vec{v} + \vec{u}\times\vec{w}$

Note: ① and ② almost determine $\vec{v} \times \vec{w}$

the rest comes from the right-hand rule







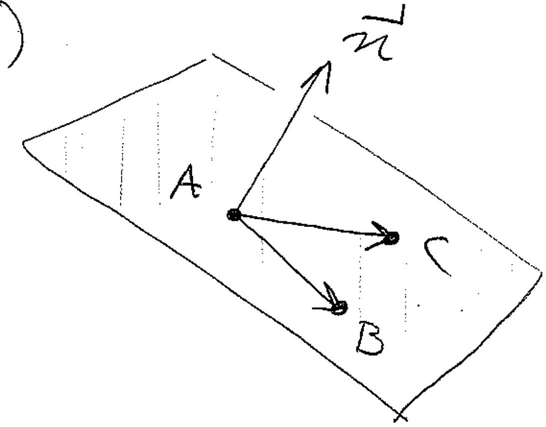
Ex: $\vec{\imath} \times \vec{j} = \vec{k}$ $\qquad \vec{j} \times \vec{k} = \vec{\imath}$

$\vec{\imath} \times \vec{k} = -\vec{j}$

Not assoc[i]ative: $\vec{\imath} \times (\vec{\imath} \times \vec{j}) = \vec{\imath} \times \vec{k} = -\vec{j}$

$(\vec{\imath} \times \vec{\imath}) \times \vec{j} = \vec{0} \times \vec{j} = \vec{0}$

Ex: Find the equation of the plane containing $A = (0,0,1)$ $\quad B = (0,1,4)$ and $C = (2,1,2)$



$\vec{v} = \overrightarrow{AB} = (0, 1, 3)$

$\vec{w} = \overrightarrow{AC} = (2, 1, 1)$

Normal vector

$\vec{n} = \vec{v} \times \vec{w} = (-2, 6, -2)$

Equation

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0 \qquad \begin{array}{l} \vec{r} = (x, y, z) \\ \vec{r}_0 = (0, 0, 1) \end{array}$$

So

$$-2x + 6y - 2z + 2 = 0.$$

Triple Product:

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = (u_1, u_2, u_3) \cdot \left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \vec{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \vec{j} + \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \vec{k} \right)$$

$$= \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} u_1 - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} u_2 + \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} u_3$$

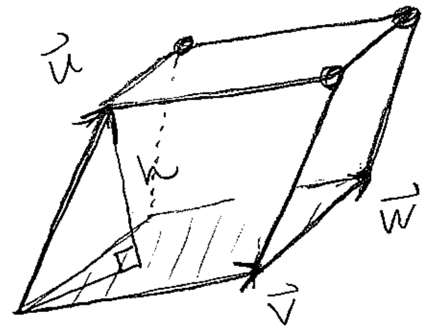
$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

This is consistient with the properties of det, cross, and dot product.

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = |\vec{u}| \, |\vec{v} \times \vec{w}| \cos\theta$$

$$= (|\vec{u}| \cos\theta)(\text{area of}$$

$$)$$

= Volume of 

$h = |\vec{u}| \cos\theta$

= (Area of base)(height)

Reasons for props of cross product (Skip)

(1) Compute $\vec{v} \cdot (\vec{v} \times \vec{w})$ and $\vec{w} \cdot (\vec{v} \times \vec{w})$ and see you get 0.

(3-5) Calculation

(2) Basically same calc as for $\left|\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right| =$ Area.
See text for details.

Q: Is there a better way to mult vectors in $\mathbb{R}^3$? (E.g. assocative, $\vec{v} \times \vec{w} \neq \vec{0}$ if $\vec{v}, \vec{w} \neq 0$)

A: No. All there is: $\mathbb{R} = \mathbb{R}^1$, $\mathbb{C} = \mathbb{R}^2$ complex numbers
$\mathbb{H} = \mathbb{R}^4$, $\mathbb{O} = \mathbb{R}^8$

Connected to the fact that when you comb a hairy sphere, there is always a whirl