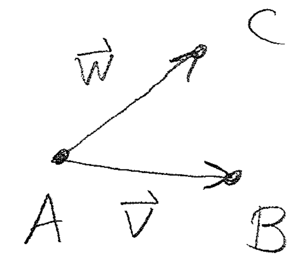
1. Consider the points $A = (0,0,2)$, $B = (1,0,3)$, and $C = (0,1,3)$ in $\mathbb{R}^3$.

(a) Compute the vectors $\mathbf{v} = \overrightarrow{AB}$ and $\mathbf{w} = \overrightarrow{AC}$. **(2 points)**

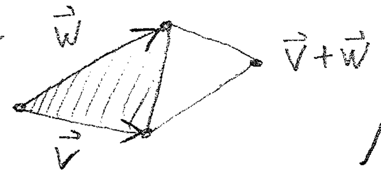


$$\vec{v} = (1,0,3) - (0,0,2) = (1,0,1)$$
$$\vec{w} = (0,1,3) - (0,0,2) = (0,1,1)$$

(b) Find a normal vector $\mathbf{n}$ to the plane P containing the points A, B, C. **(3 points)**

$$\vec{n} = \vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \vec{k}$$
$$= (-1,-1,1)$$

(c) Find the area of the triangle spanned by A, B, C. **(2 points)**

$$\text{Area}\,\Delta = \frac{1}{2}\,\text{Area}\Big($$


$$\Big) = \frac{1}{2}|\vec{v}\times\vec{w}|$$

$$= \frac{1}{2}|(-1,-1,1)| = \frac{1}{2}\sqrt{(-1)^2+(-1)^2+1^2} = \frac{\sqrt{3}}{2}$$

(d) Find an equation which describes P. If you can't do (b), take $\mathbf{n} = (1,-2,-1)$. **(1 point)**

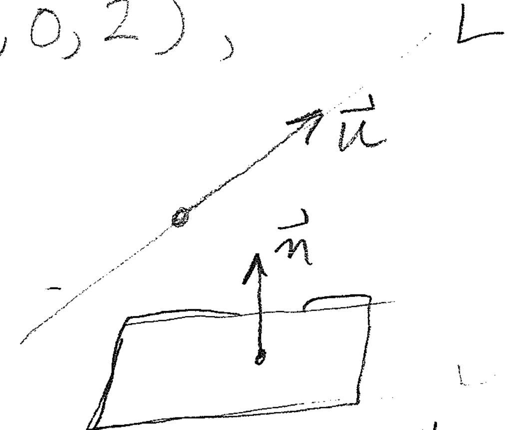
$$\vec{n} = (-1,-1,1) \qquad \text{Point} = A = (0,0,2)$$

Eqn:
$$-1(x-0) - 1(y-0) + 1(z-2) = 0 \iff$$
$$-x-y+z = 2$$

(e) Consider the line L given by the parameterization $\mathbf{r}(t) = (2+2t, 3, -1+2t)$. Is L parallel to the plane P? Why or why not? **(2 points)**

Have $r(t) = (2,3,-1) + t(2,0,2)$,

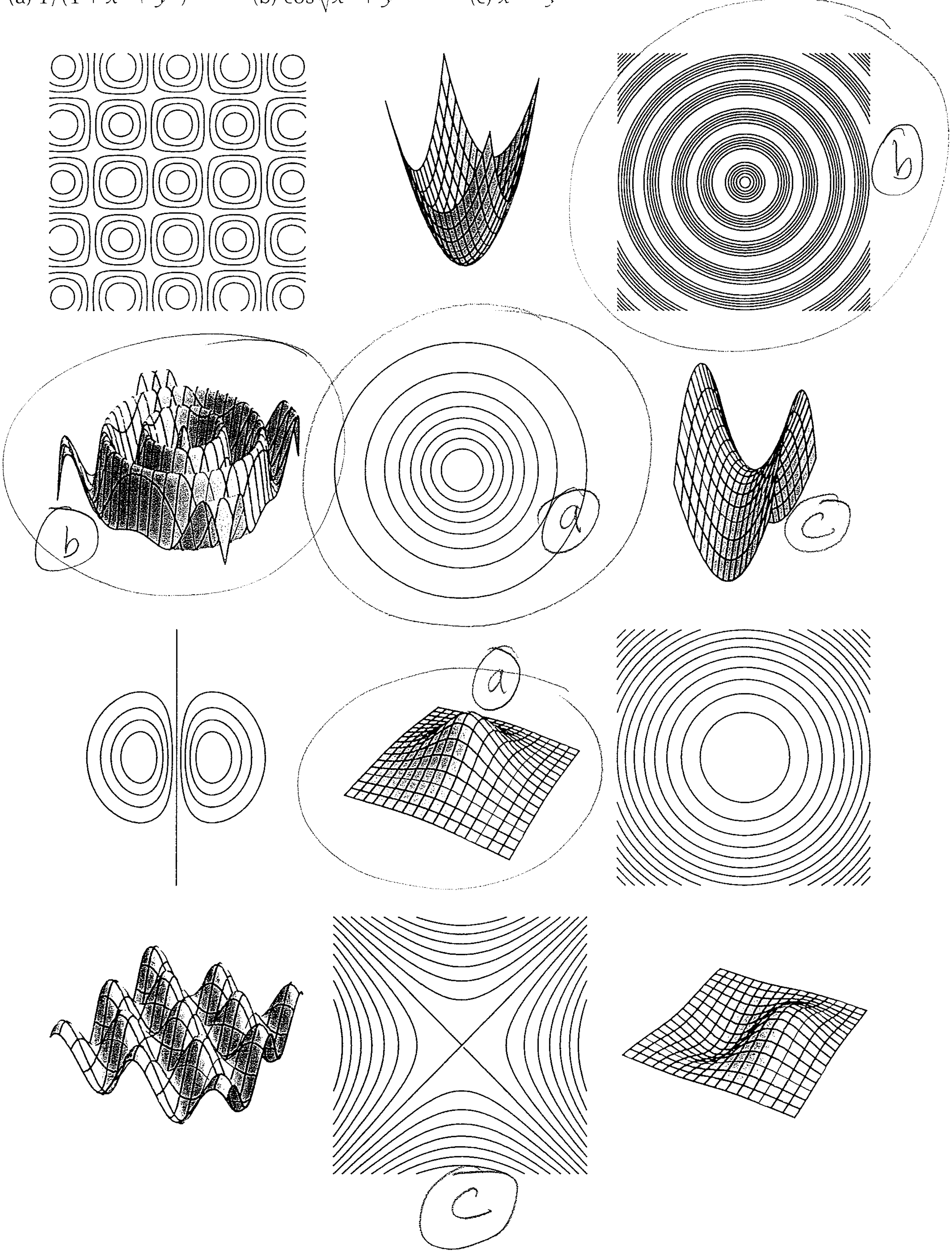
so $\vec{u} = (2,0,2)$ points along L



As
$$\vec{n}\cdot\vec{u} = (-1,-1,1)\cdot(2,0,2)$$
$$= -2+0+2 = 0,$$ the line L is parallel to P.

2. Match the following functions with their graphs and level set diagrams. Here each level set diagram consists of level sets $\{f(\mathbf{x}) = c_i\}$ drawn for evenly spaced c_i. **(9 point)**

(a) $1/(1 + x^2 + y^2)$ (b) $\cos\sqrt{x^2 + y^2}$ (c) $x^2 - y^2$



3. Consider the function $f(x,y) = \dfrac{y^2}{x^2+y^2}$ for $(x,y) \neq (0,0)$. Compute the following limit, if it exists. **(5 points)**

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

Along the x-axis, $f(x,0) = \dfrac{0^2}{x^2+0^2} = 0$

and along the y-axis we have

$f(0,y) = \dfrac{y^2}{0^2+y^2} = 1$. Thus f approaches different values depending on how we approach (0,0). So the limit D.N.E.

4. Consider the composition of the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ with $x, y\colon \mathbb{R}^2 \to \mathbb{R}$, that is

$$h(s,t) = f\big(x(s,t), y(s,t)\big)$$

Compute $\dfrac{\partial h}{\partial s}(1,2)$ using the chain rule and the table of values below. **(5 points)**

input	x	y	f	$\frac{\partial x}{\partial s}$	$\frac{\partial y}{\partial s}$	$\frac{\partial f}{\partial x}$	$\frac{\partial f}{\partial y}$
(0,1)	1	1	4	1	2	7	3
(1,1)	1	2	6	1	1	6	2
(1,2)	0	1	5	2	3	5	1
(2,3)	2	3	4	0	1	4	1

Know

$$\frac{\partial h}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}$$

Thus

$$\frac{\partial h}{\partial s}(1,2) = \frac{\partial f}{\partial x}\big(\overset{=0}{x(1,2)}, \overset{=1}{y(1,2)}\big)\cdot\frac{\partial x}{\partial s}(1,2) + \frac{\partial f}{\partial y}\big(x(1,2), y(1,2)\big)\cdot\frac{\partial y}{\partial s}(1,2)$$

$$= 7\cdot 2 + 3\cdot 3 = \boxed{23}$$

5. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = x^2 + \frac{x}{y}$.

(a) Compute the partial derivatives f_x, f_y and f_{xy}. **(3 points)**

$$f_x = 2x + \frac{1}{y} \qquad f_y = 0 - \frac{x}{y^2} = \frac{-x}{y^2}$$

$$f_{xy} = \frac{\partial}{\partial x} f_y = \frac{\partial}{\partial x}\left(-\frac{x}{y^2}\right) = -\frac{1}{y^2}$$

(b) Is f differentiable at $(2,1)$? Why or why not? **(2 points)**

Yes, because both partial derivatives $f_x = 2x + \frac{1}{y}$ and $f_y = -\frac{x^2}{y}$ exist and are continuous near $(1,2)$.

(c) Give the linear approximation of f at the point $(2,1)$: **(2 points)**

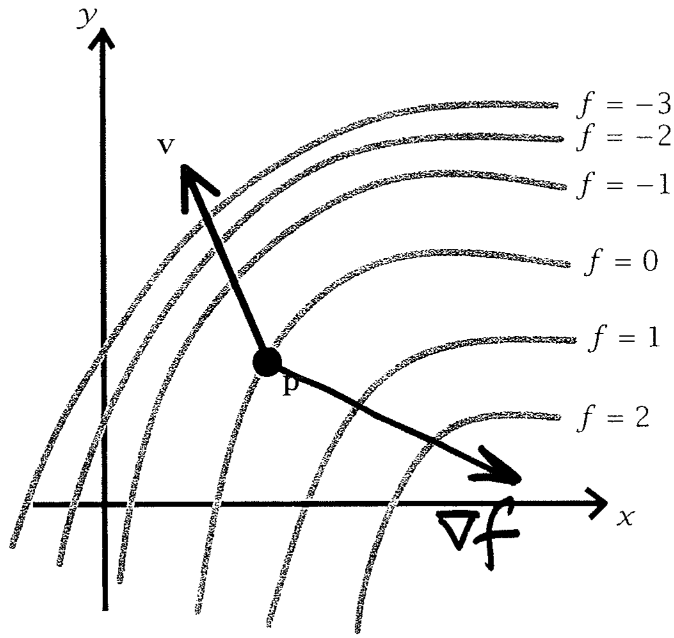
$$f(2+\Delta x, 1+\Delta y) \approx f(2,1) + f_x(2,1)\Delta x + f_y(2,1)\Delta y$$
$$= 6 + 5\Delta x - 2\Delta y$$

(d) Give the equation of the tangent plane to the graph of f at $(2,1,6)$. **(2 points)**

As $f(x,y) \simeq 6 + 5(x-2) - 2(y-1)$ by (c), tangent plane is

$$z = 6 + 5(x-2) - 2(y-1) = -2 + 5x - 2y$$

6. The picture below shows some level sets of a function $f\colon \mathbb{R}^2 \to \mathbb{R}$.



(a) At the point $\mathbf{p}$ shown, determine the sign of each of the below quantities. **(1 points each)**

$f(\mathbf{p})$: positive negative $\boxed{0}$

$f_y(\mathbf{p})$: positive $\boxed{\text{negative}}$ 0

$D_{\mathbf{v}}f(\mathbf{p})$: positive $\boxed{\text{negative}}$ 0

$f_x(\mathbf{p})$: $\boxed{\text{positive}}$ negative 0

$f_{xx}(\mathbf{p})$: positive $\boxed{\text{negative}}$ 0

(b) Draw $\nabla f(\mathbf{p})$ on the picture **(1 points)**.

Extra credit problem: Let $E\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $E(x,y) = 3x^2 + xy$. Find a $\delta > 0$ so that $|E(\mathbf{h})| < 0.01$ for all $\mathbf{h} = (x,y)$ with $|\mathbf{h}| < \delta$. Carefully justify why the δ you provide is good enough. **(3 points)**

Take $\delta = \frac{1}{100}$. If $|\vec{h}| < \delta$, then

$|x| < \delta$ and $|y| < \delta$

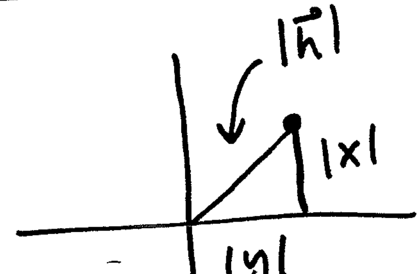
as well. Then

$$|E(\vec{h})| = |3x^2 + xy|$$

$$\leq |3x^2| + |xy| = 3|x|^2 + |x||y|$$

$$< 3\delta^2 + \delta\cdot\delta = 4\delta^2 = \frac{4}{10{,}000} < \frac{1}{10},$$

as requested!



hypotenuse is longest side.