Section 15.4

1. The region R is more easily described by polar coordinates: $R = \left\{(r,\theta) \mid 0 \le r \le 4, 0 \le \theta \le \frac{3\pi}{2}\right\}$.

Thus $\iint_R f(x,y)\,dA = \int_0^{3\pi/2}\int_0^4 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

2. The region R is more easily described by rectangular coordinates: $R = \left\{(x,y) \mid -1 \le x \le 1, 0 \le y \le 1 - x^2\right\}$.

Thus $\iint_R f(x,y)\,dA = \int_{-1}^1 \int_0^{1-x^2} f(x,y)\,dy\,dx$.

3. The region R is more easily described by rectangular coordinates: $R = \left\{(x,y) \mid -1 \le x \le 1, 0 \le y \le \frac{1}{2}x + \frac{1}{2}\right\}$.

Thus $\iint_R f(x,y)\,dA = \int_{-1}^1 \int_0^{(x+1)/2} f(x,y)\,dy\,dx$.

4. The region R is more easily described by polar coordinates: $R = \left\{(r,\theta) \mid 3 \le r \le 6, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right\}$.

Thus $\iint_R f(x,y)\,dA = \int_{-\pi/2}^{\pi/2}\int_3^6 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

36. (a) $\iint_{D_a} e^{-(x^2+y^2)}\,dA = \int_0^{2\pi}\int_0^a re^{-r^2}\,dr\,d\theta = 2\pi\left[-\frac{1}{2}e^{-r^2}\right]_0^a = \pi\left(1 - e^{-a^2}\right)$ for each a. Then $\lim_{a\to\infty}\pi\left(1 - e^{-a^2}\right) = \pi$ since $e^{-a^2} \to 0$ as $a \to \infty$. Hence $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dA = \pi$.

(b) $\iint_{S_a} e^{-(x^2+y^2)}\,dA = \int_{-a}^a\int_{-a}^a e^{-x^2}e^{-y^2}\,dx\,dy = \left(\int_{-a}^a e^{-x^2}\,dx\right)\left(\int_{-a}^a e^{-y^2}\,dy\right)$ for each a.

Then, from (a), $\pi = \iint_{\mathbb{R}^2} -(x^2+y^2)\,dA$, so

$\pi = \lim_{a\to\infty}\iint_{S_a} e^{-(x^2+y^2)}\,dA = \lim_{a\to\infty}\left(\int_{-a}^a e^{-x^2}\,dx\right)\left(\int_{-a}^a e^{-y^2}\,dy\right) = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right)$.

To evaluate $\lim_{a\to\infty}\left(\int_{-a}^a e^{-x^2}\,dx\right)\left(\int_{-a}^a e^{-y^2}\,dy\right)$, we are using the fact that these integrals are bounded. This is true since on $[-1,1]$, $0 < e^{-x^2} \le 1$ while on $(-\infty,-1)$, $0 < e^{-x^2} \le e^x$ and on $(1,\infty)$, $0 < e^{-x^2} < e^{-x}$. Hence

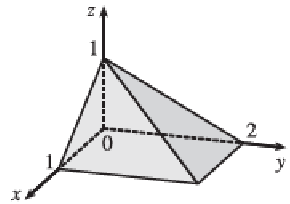
$0 \le \int_{-\infty}^{\infty} e^{-x^2}\,dx \le \int_{-\infty}^{-1} e^x\,dx + \int_{-1}^1 dx + \int_1^{\infty} e^{-x}\,dx = 2(e^{-1}+1)$.

(c) Since $\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) = \pi$ and y can be replaced by x, $\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \pi$ implies that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \pm\sqrt{\pi}$. But $e^{-x^2} \ge 0$ for all x, so $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

(d) Letting $t = \sqrt{2}\,x$, $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2}}\left(e^{-t^2/2}\right)dt$, so that $\sqrt{\pi} = \frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} e^{-t^2/2}\,dt$ or $\int_{-\infty}^{\infty} e^{-t^2/2}\,dt = \sqrt{2\pi}$.

Section 15.6

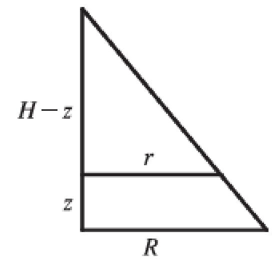
27. $E = \{(x,y,z) \mid 0 \le x \le 1, 0 \le z \le 1 - x, 0 \le y \le 2 - 2z\}$, the solid bounded by the three coordinate planes and the planes $z = 1 - x$, $y = 2 - 2z$.



Section 15.7

29. (a) The mountain comprises a solid conical region C. The work done in lifting a small volume of material ΔV with density $g(P)$ to a height $h(P)$ above sea level is $h(P)g(P)\,\Delta V$. Summing over the whole mountain we get $W = \iiint_C h(P)g(P)\,dV$.

(b) Here C is a solid right circular cone with radius $R = 62{,}000$ ft, height $H = 12{,}400$ ft, and density $g(P) = 200\ \text{lb/ft}^3$ at all points P in C. We use cylindrical coordinates:



$$\frac{r}{R} = \frac{H-z}{H} = 1 - \frac{z}{H}$$

$$W = \int_0^{2\pi}\int_0^H\int_0^{R(1-z/H)} z\cdot 200r\,dr\,dz\,d\theta = 2\pi\int_0^H 200z\left[\tfrac{1}{2}r^2\right]_{r=0}^{r=R(1-z/H)}dz$$

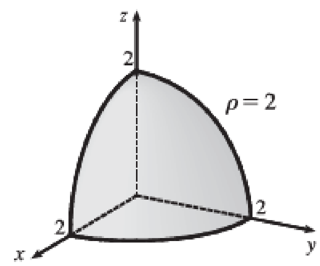
$$= 400\pi\int_0^H z\,\frac{R^2}{2}\left(1-\frac{z}{H}\right)^2 dz = 200\pi R^2\int_0^H\left(z-\frac{2z^2}{H}+\frac{z^3}{H^2}\right)dz$$

$$= 200\pi R^2\left[\frac{z^2}{2}-\frac{2z^3}{3H}+\frac{z^4}{4H^2}\right]_0^H = 200\pi R^2\left(\frac{H^2}{2}-\frac{2H^2}{3}+\frac{H^2}{4}\right)$$

$$= \tfrac{50}{3}\pi R^2H^2 = \tfrac{50}{3}\pi(62{,}000)^2(12{,}400)^2 \approx 3.1\times 10^{19}\ \text{ft-lb}$$

Section 15.8

11. $\rho = 2$ represents a sphere of radius 2, centered at the origin, so $\rho \le 2$ is this sphere and its interior. $0 \le \phi \le \frac{\pi}{2}$ restricts the solid to that portion of the region that lies on or above the xy-plane, and $0 \le \theta \le \frac{\pi}{2}$ further restricts the solid to the first octant. Thus the solid is the portion in the first octant of the solid ball centered at the origin with radius 2.



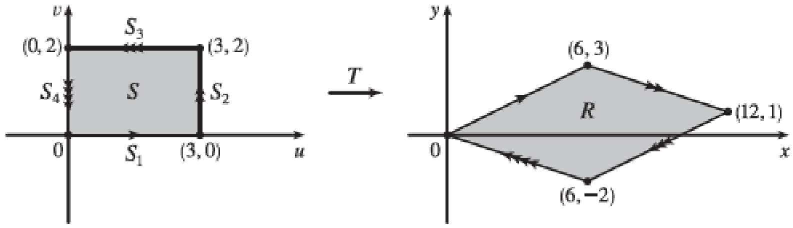
28. If we center the ball at the origin, then the ball is given by $B = \{(\rho,\theta,\phi) \mid 0 \le \rho \le a, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}$ and the distance from any point (x,y,z) in the ball to the center $(0,0,0)$ is $\sqrt{x^2+y^2+z^2} = \rho$. Thus the average distance is

$$\frac{1}{V(B)}\iiint_B \rho\,dV = \frac{1}{\frac{4}{3}\pi a^3}\int_0^\pi\int_0^{2\pi}\int_0^a \rho\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \frac{3}{4\pi a^3}\int_0^\pi \sin\phi\,d\phi\int_0^{2\pi}d\theta\int_0^a \rho^3\,d\rho$$

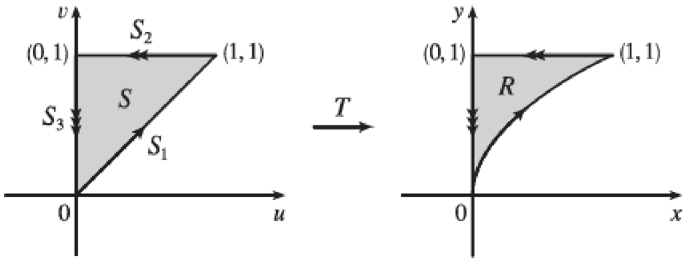
$$= \frac{3}{4\pi a^3}\left[-\cos\phi\right]_0^\pi\ \left[\theta\right]_0^{2\pi}\ \left[\tfrac{1}{4}\rho^4\right]_0^a = \frac{3}{4\pi a^3}(2)(2\pi)\left(\tfrac{1}{4}a^4\right) = \tfrac{3}{4}a$$

Section 15.9

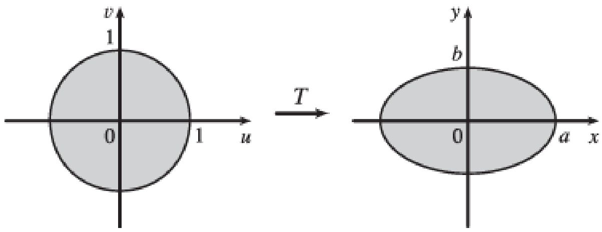
7. The transformation maps the boundary of S to the boundary of the image R, so we first look at side S_1 in the uv-plane. S_1 is described by $v = 0 \quad [0 \le u \le 3]$, so $x = 2u + 3v = 2u$ and $y = u - v = u$. Eliminating u, we have $x = 2y$, $0 \le x \le 6$. S_2 is the line segment $u = 3$, $0 \le v \le 2$, so $x = 6 + 3v$ and $y = 3 - v$. Then $v = 3 - y \quad \Rightarrow \quad x = 6 + 3(3 - y) = 15 - 3y$, $6 \le x \le 12$. S_3 is the line segment $v = 2$, $0 \le u \le 3$, so $x = 2u + 6$ and $y = u - 2$, giving $u = y + 2 \quad \Rightarrow \quad x = 2y + 10$, $6 \le x \le 12$. Finally, S_4 is the segment $u = 0$, $0 \le v \le 2$, so $x = 3v$ and $y = -v \quad \Rightarrow \quad x = -3y$, $0 \le x \le 6$. The image of set S is the region R shown in the xy-plane, a parallelogram bounded by these four segments.



9. S_1 is the line segment $u = v$, $0 \le u \le 1$, so $y = v = u$ and $x = u^2 = y^2$. Since $0 \le u \le 1$, the image is the portion of the parabola $x = y^2$, $0 \le y \le 1$. S_2 is the segment $v = 1$, $0 \le u \le 1$, thus $y = v = 1$ and $x = u^2$, so $0 \le x \le 1$. The image is the line segment $y = 1$, $0 \le x \le 1$. S_3 is the segment $u = 0$, $0 \le v \le 1$, so $x = u^2 = 0$ and $y = v \quad \Rightarrow \quad 0 \le y \le 1$. The image is the segment $x = 0$, $0 \le y \le 1$. Thus, the image of S is the region R in the first quadrant bounded by the parabola $x = y^2$, the y-axis, and the line $y = 1$.



10. Substituting $u = \dfrac{x}{a}$, $v = \dfrac{y}{b}$ into $u^2 + v^2 \le 1$ gives $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \le 1$, so the image of $u^2 + v^2 \le 1$ is the elliptical region $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \le 1$.



17. (a) $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$ and since $u = \dfrac{x}{a}$, $v = \dfrac{y}{b}$, $w = \dfrac{z}{c}$ the solid enclosed by the ellipsoid is the image of the ball $u^2 + v^2 + w^2 \le 1$. So

$$\iiint_E dV = \iiint_{u^2+v^2+w^2 \le 1} abc\, du\, dv\, dw = (abc)(\text{volume of the ball}) = \tfrac{4}{3}\pi abc$$

(b) If we approximate the surface of the earth by the ellipsoid $\dfrac{x^2}{6378^2} + \dfrac{y^2}{6378^2} + \dfrac{z^2}{6356^2} = 1$, then we can estimate the volume of the earth by finding the volume of the solid E enclosed by the ellipsoid. From part (a), this is $\iiint_E dV = \frac{4}{3}\pi(6378)(6378)(6356) \approx 1.083 \times 10^{12}\ \text{km}^3$.

18. The moment of intertia about the z-axis is $I_z = \iiint_E \left(x^2 + y^2\right) \rho(x,y,z)\, dV$, where E is the solid enclosed by $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$. As in Exercise 17(a), we use the transformation $x = au$, $y = bv$, $z = cw$, so $\left|\dfrac{\partial(x,y,z)}{\partial(u,v,w)}\right| = abc$ and

$$I_z = \iiint_E \left(x^2+y^2\right) k\, dV = \iiint_{u^2+v^2+w^2 \le 1} k(a^2u^2 + b^2v^2)(abc)\, du\, dv\, dw$$

$$= abck \int_0^\pi \int_0^{2\pi} \int_0^1 (a^2\rho^2 \sin^2\phi \cos^2\theta + b^2\rho^2 \sin^2\phi \sin^2\theta)\, \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$

$$= abck \left[a^2 \int_0^\pi \int_0^{2\pi} \int_0^1 (\rho^2 \sin^2\phi \cos^2\theta)\, \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi + b^2 \int_0^\pi \int_0^{2\pi} \int_0^1 (\rho^2 \sin^2\phi \sin^2\theta)\, \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi\right]$$

$$= a^3bck \int_0^\pi \sin^3\phi\, d\phi \int_0^{2\pi} \cos^2\theta\, d\theta \int_0^1 \rho^4\, d\rho + ab^3ck \int_0^\pi \sin^3\phi\, d\phi \int_0^{2\pi} \sin^2\theta\, d\theta \int_0^1 \rho^4\, d\rho$$

$$= a^3bck \left[\tfrac{1}{3}\cos^3\phi - \cos\phi\right]_0^\pi \left[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} \left[\tfrac{1}{5}\rho^5\right]_0^1 + ab^3ck \left[\tfrac{1}{3}\cos^3\phi - \cos\phi\right]_0^\pi \left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} \left[\tfrac{1}{5}\rho^5\right]_0^1$$

$$= a^3bck \left(\tfrac{4}{3}\right)(\pi)\left(\tfrac{1}{5}\right) + ab^3ck \left(\tfrac{4}{3}\right)(\pi)\left(\tfrac{1}{5}\right) = \tfrac{4}{15}\pi\left(a^2+b^2\right) abck$$

24. Let $u = x + y$ and $v = y$, then $x = u - v$, $y = v$, $\dfrac{\partial(x,y)}{\partial(u,v)} = 1$ and R is the image under T of the triangular region with vertices $(0,0)$, $(1,0)$ and $(1,1)$. Thus

$$\iint_R f(x+y)\, dA = \int_0^1 \int_0^u (1)\, f(u)\, dv\, du = \int_0^1 f(u)\left[\,v\,\right]_{v=0}^{v=u} du = \int_0^1 u f(u)\, du \quad \text{as desired.}$$

Section 15 Review

49. Let $u = y - x$ and $v = y + x$ so $x = y - u = (v - x) - u \quad \Rightarrow \quad x = \frac{1}{2}(v - u)$ and $y = v - \frac{1}{2}(v - u) = \frac{1}{2}(v + u)$.

$\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = \left|\dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial x}{\partial v}\dfrac{\partial y}{\partial u}\right| = \left|-\frac{1}{2}\left(\frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{2}\right)\right| = \left|-\frac{1}{2}\right| = \frac{1}{2}$. R is the image under this transformation of the square with vertices $(u,v) = (0,0)$, $(-2,0)$, $(0,2)$, and $(-2,2)$. So

$$\iint_R xy\, dA = \int_0^2 \int_{-2}^0 \frac{v^2 - u^2}{4}\left(\frac{1}{2}\right) du\, dv = \tfrac{1}{8}\int_0^2 \left[v^2u - \tfrac{1}{3}u^3\right]_{u=-2}^{u=0} dv = \tfrac{1}{8}\int_0^2 \left(2v^2 - \tfrac{8}{3}\right) dv = \tfrac{1}{8}\left[\tfrac{2}{3}v^3 - \tfrac{8}{3}v\right]_0^2 = 0$$

This result could have been anticipated by symmetry, since the integrand is an odd function of y and R is symmetric about the x-axis.

Section 16.4

22. By Green's Theorem, $\frac{1}{2A}\oint_C x^2\,dy = \frac{1}{2A}\iint_D 2x\,dA = \frac{1}{A}\iint_D x\,dA = \overline{x}$ and

$-\frac{1}{2A}\oint_C y^2\,dx = -\frac{1}{2A}\iint_D(-2y)\,dA = \frac{1}{A}\iint_D y\,dA = \overline{y}$.

27. Since C is a simple closed path which doesn't pass through or enclose the origin, there exists an open region that doesn't contain the origin but does contain D. Thus $P = -y/(x^2+y^2)$ and $Q = x/(x^2+y^2)$ have continuous partial derivatives on this open region containing D and we can apply Green's Theorem. But by Exercise 17.3.33(a) [ET 16.3.33(a)], $\partial P/\partial y = \partial Q/\partial x$, so $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_D \mathbf{0}\,dA = \mathbf{0}$.

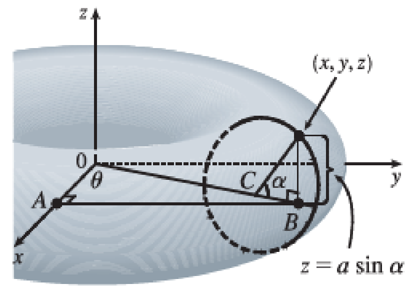
Section 16.6

60. (a) Here $z = a\sin\alpha$, $y = |AB|$, and $x = |OA|$. But

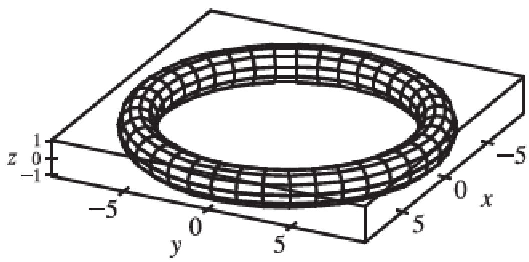
$|OB| = |OC| + |CB| = b + a\cos\alpha$ and $\sin\theta = \dfrac{|AB|}{|OB|}$ so that

$y = |OB|\sin\theta = (b + a\cos\alpha)\sin\theta$. Similarly $\cos\theta = \dfrac{|OA|}{|OB|}$ so

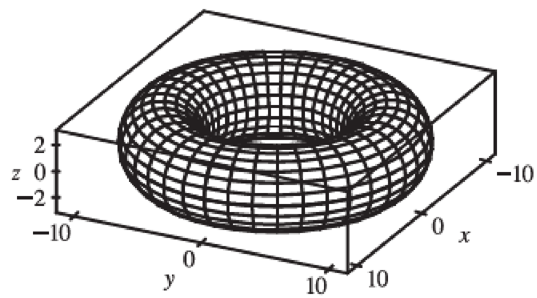
$x = (b + a\cos\alpha)\cos\theta$. Hence a parametric representation for the torus is $x = b\cos\theta + a\cos\alpha\,\cos\theta$, $y = b\sin\theta + a\cos\alpha\,\sin\theta$, $z = a\sin\alpha$, where $0 \le \alpha \le 2\pi$, $0 \le \theta \le 2\pi$.



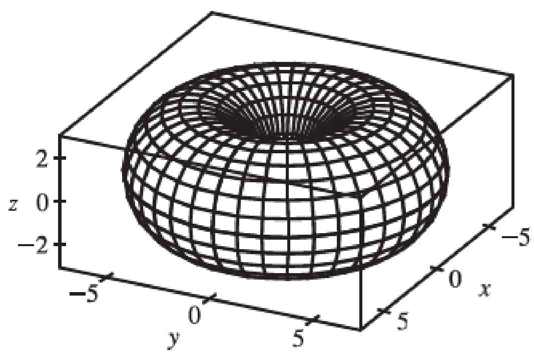
(b)



$a = 1, b = 8$



$a = 3, b = 8$



$a = 3, b = 4$

(c) $x = b\cos\theta + a\cos\alpha\,\cos\theta$, $y = b\sin\theta + a\cos\alpha\,\sin\theta$, $z = a\sin\alpha$, so $\mathbf{r}_\alpha = \langle -a\sin\alpha\,\cos\theta, -a\sin\alpha\,\sin\theta, a\cos\alpha\rangle$,

$\mathbf{r}_\theta = \langle -(b + a\cos\alpha)\sin\theta, (b + a\cos\alpha)\cos\theta, 0\rangle$ and

$$\begin{aligned}\mathbf{r}_\alpha \times \mathbf{r}_\theta &= \left(-ab\cos\alpha\,\cos\theta - a^2\cos\alpha\,\cos^2\theta\right)\mathbf{i} + \left(-ab\sin\alpha\,\cos\theta - a^2\sin\alpha\,\cos^2\theta\right)\mathbf{j} \\ &\qquad + \left(-ab\cos^2\alpha\,\sin\theta - a^2\cos^2\alpha\,\sin\theta\,\cos\theta - ab\sin^2\alpha\,\sin\theta - a^2\sin^2\alpha\,\sin\theta\,\cos\theta\right)\mathbf{k} \\ &= -a\,(b + a\cos\alpha)\left[(\cos\theta\,\cos\alpha)\,\mathbf{i} + (\sin\theta\,\cos\alpha)\,\mathbf{j} + (\sin\alpha)\,\mathbf{k}\right]\end{aligned}$$

Then $|\mathbf{r}_\alpha \times \mathbf{r}_\theta| = a(b + a\cos\alpha)\sqrt{\cos^2\theta\,\cos^2\alpha + \sin^2\theta\,\cos^2\alpha + \sin^2\alpha} = a(b + a\cos\alpha)$.

Note: $b > a$, $-1 \le \cos\alpha \le 1$ so $|b + a\cos\alpha| = b + a\cos\alpha$. Hence

$A(S) = \int_0^{2\pi}\int_0^{2\pi} a(b + a\cos\alpha)\,d\alpha\,d\theta = 2\pi\left[ab\alpha + a^2\sin\alpha\right]_0^{2\pi} = 4\pi^2 ab$.