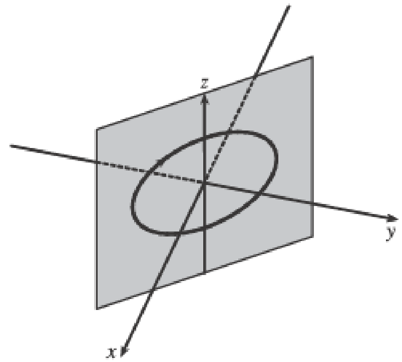
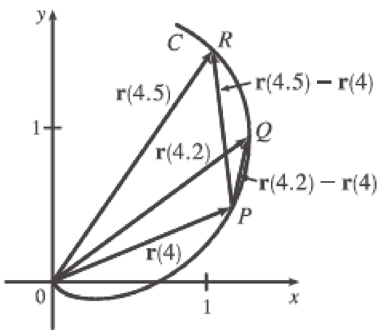
## Section 13.1

**14.** If $x = \cos t,\ y = -\cos t,\ z = \sin t$, then $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$, so the curve is contained in the intersection of circular cylinders along the $x$- and $y$-axes. Furthermore, $y = -x$, so the curve is an ellipse in the plane $y = -x$, centered at the origin.



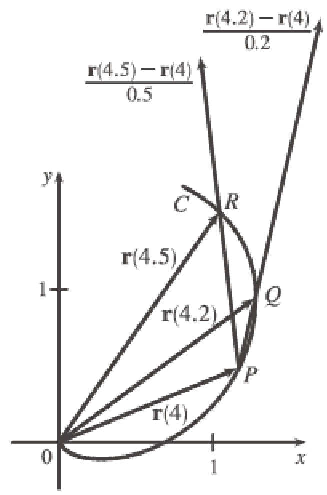

## Section 13.2

**1.** (a)



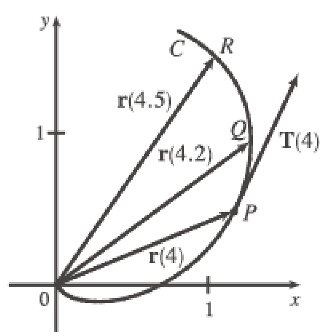

(b) $\dfrac{\mathbf{r}(4.5) - \mathbf{r}(4)}{0.5} = 2[\mathbf{r}(4.5) - \mathbf{r}(4)]$, so we draw a vector in the same direction but with twice the length of the vector $\mathbf{r}(4.5) - \mathbf{r}(4)$.

$\dfrac{\mathbf{r}(4.2) - \mathbf{r}(4)}{0.2} = 5[\mathbf{r}(4.2) - \mathbf{r}(4)]$, so we draw a vector in the same direction but with 5 times the length of the vector $\mathbf{r}(4.2) - \mathbf{r}(4)$.



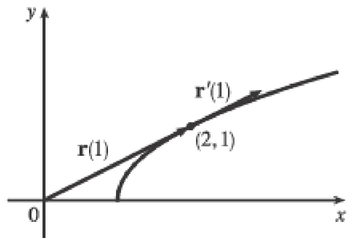

(c) By Definition 1, $\mathbf{r}'(4) = \lim_{h \to 0} \dfrac{\mathbf{r}(4 + h) - \mathbf{r}(4)}{h}$. $\mathbf{T}(4) = \dfrac{\mathbf{r}'(4)}{|\mathbf{r}'(4)|}$.

(d) $\mathbf{T}(4)$ is a unit vector in the same direction as $\mathbf{r}'(4)$, that is, parallel to the tangent line to the curve at $\mathbf{r}(4)$ with length 1.



**4.** Since $x = 1 + t = 1 + y^2$, the curve is part of a parabola. Here we have $y \geq 0$.

(a), (c)




(b) $\mathbf{r}'(t) = \left\langle 1, \dfrac{1}{2\sqrt{t}} \right\rangle$,

$\mathbf{r}'(1) = \langle 1, \frac{1}{2} \rangle$

**15.** Here $\mathbf{r}(t) = \langle 3\sin t, 4t, 3\cos t \rangle$, so $\mathbf{r}'(t) = \langle 3\cos t, 4, -3\sin t \rangle$ and $|\mathbf{r}'(t)| = \sqrt{9\cos^2 t + 16 + 9\sin^2 t} = \sqrt{25} = 5$. The point $(0, 0, 3)$ corresponds to $t = 0$, so the arc length function beginning at $(0, 0, 3)$ and measuring in the positive direction is given by $s(t) = \int_0^t |\mathbf{r}'(u)|\, du = \int_0^t 5\, du = 5t$. $s(t) = 5 \Rightarrow 5t = 5 \Rightarrow t = 1$, thus your location after moving 5 units along the curve is $(3\sin 1, 4, 3\cos 1)$.

Section 14.6

**54.** First note that the point $(1, 1, 2)$ is on both surfaces. For the ellipsoid, an equation of the tangent plane at $(1, 1, 2)$ is $6x + 4y + 4z = 18$ or $3x + 2y + 2z = 9$, and for the sphere, an equation of the tangent plane at $(1, 1, 2)$ is $(2 - 8)x + (2 - 6)y + (4 - 8)z = -18$ or $-6x - 4y - 4z = -18$ or $3x + 2y + 2z = 9$. Since these tangent planes are the same, the surfaces are tangent to each other at the point $(1, 1, 2)$.

Section 14.7

**2.** (a) $D = g_{xx}(0, 2)\, g_{yy}(0, 2) - [g_{xy}(0, 2)]^2 = (-1)(1) - (6)^2 = -37$. Since $D < 0$, $g$ has a saddle point at $(0, 2)$ by the Second Derivatives Test.

(b) $D = g_{xx}(0, 2)\, g_{yy}(0, 2) - [g_{xy}(0, 2)]^2 = (-1)(-8) - (2)^2 = 4$. Since $D > 0$ and $g_{xx}(0, 2) < 0$, $g$ has a local maximum at $(0, 2)$ by the Second Derivatives Test.

(c) $D = g_{xx}(0, 2)\, g_{yy}(0, 2) - [g_{xy}(0, 2)]^2 = (4)(9) - (6)^2 = 0$. In this case the Second Derivatives Test gives no information about $g$ at the point $(0, 2)$.

**3.** In the figure, a point at approximately $(1, 1)$ is enclosed by level curves which are oval in shape and indicate that as we move away from the point in any direction the values of $f$ are increasing. Hence we would expect a local minimum at or near $(1, 1)$. The level curves near $(0, 0)$ resemble hyperbolas, and as we move away from the origin, the values of $f$ increase in some directions and decrease in others, so we would expect to find a saddle point there.

To verify our predictions, we have $f(x, y) = 4 + x^3 + y^3 - 3xy \Rightarrow f_x(x, y) = 3x^2 - 3y$, $f_y(x, y) = 3y^2 - 3x$. We have critical points where these partial derivatives are equal to 0: $3x^2 - 3y = 0$, $3y^2 - 3x = 0$. Substituting $y = x^2$ from the first equation into the second equation gives $3(x^2)^2 - 3x = 0 \Rightarrow 3x(x^3 - 1) = 0 \Rightarrow x = 0$ or $x = 1$. Then we have two critical points, $(0, 0)$ and $(1, 1)$. The second partial derivatives are $f_{xx}(x, y) = 6x$, $f_{xy}(x, y) = -3$, and $f_{yy}(x, y) = 6y$, so $D(x, y) = f_{xx}(x, y)\, f_{yy}(x, y) - [f_{xy}(x, y)]^2 = (6x)(6y) - (-3)^2 = 36xy - 9$. Then $D(0, 0) = 36(0)(0) - 9 = -9$, and $D(1, 1) = 36(1)(1) - 9 = 27$. Since $D(0, 0) < 0$, $f$ has a saddle point at $(0, 0)$ by the Second Derivatives Test. Since $D(1, 1) > 0$ and $f_{xx}(1, 1) > 0$, $f$ has a local minimum at $(1, 1)$.

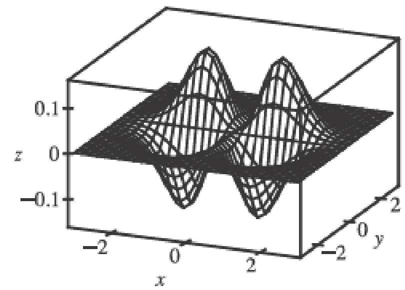
**20.** $f(x,y) = x^2ye^{-x^2-y^2} \quad \Rightarrow$

$f_x = x^2ye^{-x^2-y^2}(-2x) + 2xye^{-x^2-y^2} = 2xy(1-x^2)e^{-x^2-y^2}$,

$f_y = x^2ye^{-x^2-y^2}(-2y) + x^2e^{-x^2-y^2} = x^2(1-2y^2)e^{-x^2-y^2}$,

$f_{xx} = 2y(2x^4 - 5x^2 + 1)e^{-x^2-y^2}$,

$f_{xy} = 2x(1-x^2)(1-2y^2)e^{-x^2-y^2}$, $f_{yy} = 2x^2y(2y^2-3)e^{-x^2-y^2}$.



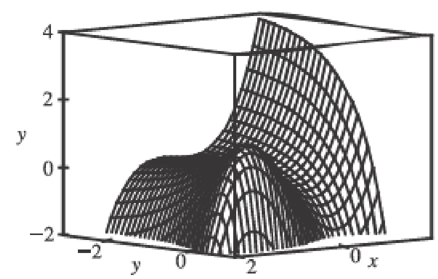

$f_x = 0$ implies $x = 0$, $y = 0$, or $x = \pm 1$. If $x = 0$ then $f_y = 0$ for any $y$-value, so all points of the form $(0, y)$ are critical points. If $y = 0$ then $f_y = 0 \quad \Rightarrow \quad x^2 e^{-x^2} = 0 \quad \Rightarrow \quad x = 0$, so $(0,0)$ (already included above) is a critical point. If $x = \pm 1$ then $(1-2y^2)e^{-1-y^2} = 0 \quad \Rightarrow \quad y = \pm\frac{1}{\sqrt{2}}$, so $\left(\pm 1, \frac{1}{\sqrt{2}}\right)$ and $\left(\pm 1, -\frac{1}{\sqrt{2}}\right)$ are critical points. Now

$D\left(\pm 1, \frac{1}{\sqrt{2}}\right) = 8e^{-3} > 0$, $f_{xx}\left(\pm 1, \frac{1}{\sqrt{2}}\right) = -2\sqrt{2}\,e^{-3/2} < 0$ and $D\left(\pm 1, -\frac{1}{\sqrt{2}}\right) = 8e^{-3} > 0$,

$f_{xx}\left(\pm 1, -\frac{1}{\sqrt{2}}\right) = 2\sqrt{2}\,e^{-3/2} > 0$, so $f\left(\pm 1, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}e^{-3/2}$ are local maximum points while

$f\left(\pm 1, -\frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}e^{-3/2}$ are local minimum points. At all critical points $(0, y)$ we have $D(0, y) = 0$, so the Second Derivatives Test gives no information. However, if $y > 0$ then $x^2ye^{-x^2-y^2} \geq 0$ with equality only when $x = 0$, so we have local minimum values $f(0, y) = 0$, $y > 0$. Similarly, if $y < 0$ then $x^2ye^{-x^2-y^2} \leq 0$ with equality when $x = 0$ so $f(0, y) = 0$, $y < 0$ are local maximum values, and $(0, 0)$ is a saddle point.

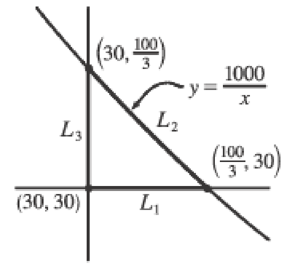
**38.** $f(x,y) = 3xe^y - x^3 - e^{3y}$ is differentiable everywhere, so the requirement for critical points is that $f_x = 3e^y - 3x^2 = 0 \quad (1)$ and $f_y = 3xe^y - 3e^{3y} = 0 \quad (2)$. From (1) we obtain $e^y = x^2$, and then (2) gives $3x^3 - 3x^6 = 0 \quad \Rightarrow \quad x = 1$ or 0, but only $x = 1$ is valid, since $x = 0$ makes (1) impossible. So substituting $x = 1$ into (1) gives $y = 0$, and the only critical point is $(1, 0)$.




The Second Derivatives Test shows that this gives a local maximum, since $D(1,0) = \left[-6x(3xe^y - 9e^{3y}) - (3e^y)^2\right]_{(1,0)} = 27 > 0$ and $f_{xx}(1,0) = [-6x]_{(1,0)} = -6 < 0$. But $f(1,0) = 1$ is not an absolute maximum because, for instance, $f(-3,0) = 17$. This can also be seen from the graph.

**52.** Let $x$ be the length of the north and south walls, $y$ the length of the east and west walls, and $z$ the height of the building. The heat loss is given by $h = 10(2yz) + 8(2xz) + 1(xy) + 5(xy) = 6xy + 16xz + 20yz$.

The volume is 4000 m$^3$, so $xyz = 4000$, and we substitute $z = \frac{4000}{xy}$ to obtain the heat loss function $h(x, y) = 6xy + 80{,}000/x + 64{,}000/y$.




(a) Since $z = \frac{4000}{xy} \geq 4$, $xy \leq 1000 \quad \Rightarrow \quad y \leq 1000/x$. Also $x \geq 30$ and $y \geq 30$, so the domain of $h$ is $D = \{(x, y) \mid x \geq 30, 30 \leq y \leq 1000/x\}$.

(b) $h(x, y) = 6xy + 80{,}000x^{-1} + 64{,}000y^{-1} \quad \Rightarrow \quad h_x = 6y - 80{,}000x^{-2}$, $h_y = 6x - 64{,}000y^{-2}$. $h_x = 0$ implies $6x^2y = 80{,}000 \quad \Rightarrow \quad y = \dfrac{80{,}000}{6x^2}$ and substituting into $h_y = 0$ gives $6x = 64{,}000\left(\dfrac{6x^2}{80{,}000}\right)^2 \quad \Rightarrow$ $x^3 = \dfrac{80{,}000^2}{6 \cdot 64{,}000} = \dfrac{50{,}000}{3}$, so $x = \sqrt[3]{\dfrac{50{,}000}{3}} = 10\sqrt[3]{\dfrac{50}{3}} \quad \Rightarrow \quad y = \dfrac{80}{\sqrt[3]{60}}$, and the only critical point of $h$ is $\left(10\sqrt[3]{\dfrac{50}{3}}, \dfrac{80}{\sqrt[3]{60}}\right) \approx (25.54, 20.43)$ which is not in $D$. Next we check the boundary of $D$.

On $L_1$: $y = 30$, $h(x, 30) = 180x + 80{,}000/x + 6400/3$, $30 \leq x \leq \frac{100}{3}$. Since $h'(x, 30) = 180 - 80{,}000/x^2 > 0$ for $30 \leq x \leq \frac{100}{3}$, $h(x, 30)$ is an increasing function with minimum $h(30, 30) = 10{,}200$ and maximum $h\left(\frac{100}{3}, 30\right) \approx 10{,}533$.

On $L_2$: $y = 1000/x$, $h(x, 1000/x) = 6000 + 64x + 80{,}000/x$, $30 \leq x \leq \frac{100}{3}$.

Since $h'(x, 1000/x) = 64 - 80{,}000/x^2 < 0$ for $30 \leq x \leq \frac{100}{3}$, $h(x, 1000/x)$ is a decreasing function with minimum $h\left(\frac{100}{3}, 30\right) \approx 10{,}533$ and maximum $h\left(30, \frac{100}{3}\right) \approx 10{,}587$.

On $L_3$: $x = 30$, $h(30, y) = 180y + 64{,}000/y + 8000/3$, $30 \leq y \leq \frac{100}{3}$. $h'(30, y) = 180 - 64{,}000/y^2 > 0$ for $30 \leq y \leq \frac{100}{3}$, so $h(30, y)$ is an increasing function of $y$ with minimum $h(30, 30) = 10{,}200$ and maximum $h\left(30, \frac{100}{3}\right) \approx 10{,}587$.

Thus the absolute minimum of $h$ is $h(30, 30) = 10{,}200$, and the dimensions of the building that minimize heat loss are walls 30 m in length and height $\frac{4000}{30^2} = \frac{40}{9} \approx 4.44$ m.

(c) From part (b), the only critical point of $h$, which gives a local (and absolute) minimum, is approximately $h(25.54, 20.43) \approx 9396$. So a building of volume 4000 m$^2$ with dimensions $x \approx 25.54$ m, $y \approx 20.43$ m, $z \approx \frac{4000}{(25.54)(20.43)} \approx 7.67$ m has the least amount of heat loss.

## Section 14.8

1. At the extreme values of $f$, the level curves of $f$ just touch the curve $g(x, y) = 8$ with a common tangent line. (See Figure 1 and the accompanying discussion.) We can observe several such occurrences on the contour map, but the level curve $f(x, y) = c$ with the largest value of $c$ which still intersects the curve $g(x, y) = 8$ is approximately $c = 59$, and the smallest value of $c$ corresponding to a level curve which intersects $g(x, y) = 8$ appears to be $c = 30$. Thus we estimate the maximum value of $f$ subject to the constraint $g(x, y) = 8$ to be about 59 and the minimum to be 30.

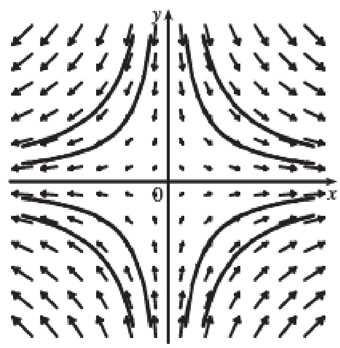
21. (a) $f(x, y) = x$, $g(x, y) = y^2 + x^4 - x^3 = 0 \Rightarrow \nabla f = \langle 1, 0 \rangle = \lambda \nabla g = \lambda \langle 4x^3 - 3x^2, 2y \rangle$. Then $1 = \lambda(4x^3 - 3x^2)$ (1) and $0 = 2\lambda y$ (2). We have $\lambda \neq 0$ from (1), so (2) gives $y = 0$. Then, from the constraint equation, $x^4 - x^3 = 0 \Rightarrow x^3(x - 1) = 0 \Rightarrow x = 0$ or $x = 1$. But $x = 0$ contradicts (1), so the only possible extreme value subject to the constraint is $f(1, 0) = 1$. (The question remains whether this is indeed the minimum of $f$.)

(b) The constraint is $y^2 + x^4 - x^3 = 0 \Leftrightarrow y^2 = x^3 - x^4$. The left side is non-negative, so we must have $x^3 - x^4 \geq 0$ which is true only for $0 \leq x \leq 1$. Therefore the minimum possible value for $f(x, y) = x$ is 0 which occurs for $x = y = 0$. However, $\lambda \nabla g(0, 0) = \lambda \langle 0 - 0, 0 \rangle = \langle 0, 0 \rangle$ and $\nabla f(0, 0) = \langle 1, 0 \rangle$, so $\nabla f(0, 0) \neq \lambda \nabla g(0, 0)$ for all values of $\lambda$.

(c) Here $\nabla g(0, 0) = \mathbf{0}$ but the method of Lagrange multipliers requires that $\nabla g \neq \mathbf{0}$ everywhere on the constraint curve.

## Section 16.1

35. (a) We sketch the vector field $\mathbf{F}(x, y) = x\,\mathbf{i} - y\,\mathbf{j}$ along with several approximate flow lines. The flow lines appear to be hyperbolas with shape similar to the graph of $y = \pm 1/x$, so we might guess that the flow lines have equations $y = C/x$.




(b) If $x = x(t)$ and $y = y(t)$ are parametric equations of a flow line, then the velocity vector of the flow line at the point $(x, y)$ is $x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j}$. Since the velocity vectors coincide with the vectors in the vector field, we have $x'(t)\,\mathbf{i} + y'(t)\,\mathbf{j} = x\,\mathbf{i} - y\,\mathbf{j} \Rightarrow dx/dt = x$, $dy/dt = -y$. To solve these differential equations, we know $dx/dt = x \Rightarrow dx/x = dt \Rightarrow \ln|x| = t + C \Rightarrow x = \pm e^{t+C} = Ae^t$ for some constant $A$, and $dy/dt = -y \Rightarrow dy/y = -dt \Rightarrow \ln|y| = -t + K \Rightarrow y = \pm e^{-t+K} = Be^{-t}$ for some constant $B$. Therefore $xy = Ae^t Be^{-t} = AB =$ constant. If the flow line passes through $(1, 1)$ then $(1)(1) =$ constant $= 1 \Rightarrow xy = 1 \Rightarrow y = 1/x$, $x > 0$.

## Section 16.2

17. (a) Along the line $x = -3$, the vectors of $\mathbf{F}$ have positive $y$-components, so since the path goes upward, the integrand $\mathbf{F} \cdot \mathbf{T}$ is always positive. Therefore $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathbf{T}\, ds$ is positive.

(b) All of the (nonzero) field vectors along the circle with radius 3 are pointed in the clockwise direction, that is, opposite the direction to the path. So $\mathbf{F} \cdot \mathbf{T}$ is negative, and therefore $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot \mathbf{T}\, ds$ is negative.

**37.** From Example 3, $\rho(x,y) = k(1-y)$, $x = \cos t$, $y = \sin t$, and $ds = dt$, $0 \le t \le \pi$ $\Rightarrow$

$$I_x = \int_C y^2 \rho(x,y)\,ds = \int_0^\pi \sin^2 t\,[k(1-\sin t)]\,dt = k\int_0^\pi (\sin^2 t - \sin^3 t)\,dt$$

$$= \tfrac{1}{2}k\int_0^\pi (1-\cos 2t)\,dt - k\int_0^\pi (1-\cos^2 t)\sin t\,dt \quad \begin{bmatrix}\text{Let } u = \cos t,\ du = -\sin t\,dt \\ \text{in the second integral}\end{bmatrix}$$

$$= k\left[\tfrac{\pi}{2} + \int_1^{-1}(1-u^2)\,du\right] = k\left(\tfrac{\pi}{2} - \tfrac{4}{3}\right)$$

$$I_y = \int_C x^2\rho(x,y)\,ds = k\int_0^\pi \cos^2 t\,(1-\sin t)\,dt = \tfrac{k}{2}\int_0^\pi (1+\cos 2t)\,dt - k\int_0^\pi \cos^2 t\sin t\,dt$$

$= k\left(\frac{\pi}{2} - \frac{2}{3}\right)$, using the same substitution as above.

**48.** Use the orientation pictured in the figure. Then since $\mathbf{B}$ is tangent to any circle that lies in the plane perpendicular to the wire, $\mathbf{B} = |\mathbf{B}|\,\mathbf{T}$ where $\mathbf{T}$ is the unit tangent to the circle $C$: $x = r\cos\theta$, $y = r\sin\theta$. Thus $\mathbf{B} = |\mathbf{B}|\,\langle -\sin\theta, \cos\theta\rangle$. Then $\int_C \mathbf{B}\cdot d\mathbf{r} = \int_0^{2\pi} |\mathbf{B}|\,\langle -\sin\theta,\cos\theta\rangle \cdot \langle -r\sin\theta, r\cos\theta\rangle\,d\theta = \int_0^{2\pi}|\mathbf{B}|\,r\,d\theta = 2\pi r\,|\mathbf{B}|$. (Note that $|\mathbf{B}|$ here is the magnitude of the field at a distance $r$ from the wire's center.) But by Ampere's Law $\int_C \mathbf{B}\cdot d\mathbf{r} = \mu_0 I$. Hence $|\mathbf{B}| = \mu_0 I/(2\pi r)$.

## Section 16.3

**11.** (a) $\mathbf{F}$ has continuous first-order partial derivatives and $\dfrac{\partial}{\partial y}2xy = 2x = \dfrac{\partial}{\partial x}(x^2)$ on $\mathbb{R}^2$, which is open and simply-connected. Thus, $\mathbf{F}$ is conservative by Theorem 6. Then we know that the line integral of $\mathbf{F}$ is independent of path; in particular, the value of $\int_C \mathbf{F}\cdot d\mathbf{r}$ depends only on the endpoints of $C$. Since all three curves have the same initial and terminal points, $\int_C \mathbf{F}\cdot d\mathbf{r}$ will have the same value for each curve.

(b) We first find a potential function $f$, so that $\nabla f = \mathbf{F}$. We know $f_x(x,y) = 2xy$ and $f_y(x,y) = x^2$. Integrating $f_x(x,y)$ with respect to $x$, we have $f(x,y) = x^2y + g(y)$. Differentiating both sides with respect to $y$ gives $f_y(x,y) = x^2 + g'(y)$, so we must have $x^2 + g'(y) = x^2 \quad\Rightarrow\quad g'(y) = 0 \quad\Rightarrow\quad g(y) = K$, a constant.
Thus $f(x,y) = x^2y + K$. All three curves start at $(1,2)$ and end at $(3,2)$, so by Theorem 2, $\int_C \mathbf{F}\cdot d\mathbf{r} = f(3,2) - f(1,2) = 18 - 2 = 16$ for each curve.

**20.** Here $\mathbf{F}(x,y) = (1-ye^{-x})\,\mathbf{i} + e^{-x}\,\mathbf{j}$. Then $f(x,y) = x + ye^{-x}$ is a potential function for $\mathbf{F}$, that is, $\nabla f = \mathbf{F}$ so $\mathbf{F}$ is conservative and thus its line integral is independent of path. Hence $\int_C (1-ye^{-x})\,dx + e^{-x}\,dy = \int_C \mathbf{F}\cdot d\mathbf{r} = f(1,2) - f(0,1) = (1+2e^{-1}) - 1 = 2/e$.

**23.** We know that if the vector field (call it $\mathbf{F}$) is conservative, then around any closed path $C$, $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$. But take $C$ to be a circle centered at the origin, oriented counterclockwise. All of the field vectors that start on $C$ are roughly in the direction of motion along $C$, so the integral around $C$ will be positive. Therefore the field is not conservative.

**24.** If a vector field $\mathbf{F}$ is conservative, then around any closed path $C$, $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$. For any closed path we draw in the field, it appears that some vectors on the curve point in approximately the same direction as the curve and a similar number point in roughly the opposite direction. (Some appear perpendicular to the curve as well.) Therefore it is plausible that $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$ for every closed curve $C$ which means $\mathbf{F}$ is conservative.

**26.** $\nabla f(x,y) = \cos(x-2y)\,\mathbf{i} - 2\cos(x-2y)\,\mathbf{j}$

(a) We use Theorem 2: $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$ where $C_1$ starts at $t = a$ and ends at $t = b$. So because $f(0,0) = \sin 0 = 0$ and $f(\pi,\pi) = \sin(\pi - 2\pi) = 0$, one possible curve $C_1$ is the straight line from $(0,0)$ to $(\pi,\pi)$; that is, $\mathbf{r}(t) = \pi t\,\mathbf{i} + \pi t\,\mathbf{j},\ 0 \le t \le 1$.

(b) From (a), $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$. So because $f(0,0) = \sin 0 = 0$ and $f\left(\frac{\pi}{2}, 0\right) = 1$, one possible curve $C_2$ is $\mathbf{r}(t) = \frac{\pi}{2}t\,\mathbf{i}, 0 \le t \le 1$, the straight line from $(0,0)$ to $\left(\frac{\pi}{2}, 0\right)$.

**33.** (a) $P = -\dfrac{y}{x^2+y^2}$, $\dfrac{\partial P}{\partial y} = \dfrac{y^2 - x^2}{(x^2+y^2)^2}$ and $Q = \dfrac{x}{x^2+y^2}$, $\dfrac{\partial Q}{\partial x} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$. Thus $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

(b) $C_1$: $x = \cos t$, $y = \sin t$, $0 \le t \le \pi$, $C_2$: $x = \cos t$, $y = \sin t$, $t = 2\pi$ to $t = \pi$. Then

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi \frac{(-\sin t)(-\sin t) + (\cos t)(\cos t)}{\cos^2 t + \sin^2 t}\, dt = \int_0^\pi dt = \pi \text{ and } \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{2\pi}^{\pi} dt = -\pi$$

Since these aren't equal, the line integral of $\mathbf{F}$ isn't independent of path. (Or notice that $\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} dt = 2\pi$ where $C_3$ is the circle $x^2 + y^2 = 1$, and apply the contrapositive of Theorem 3.) This doesn't contradict Theorem 6, since the domain of $\mathbf{F}$, which is $\mathbb{R}^2$ except the origin, isn't simply-connected.