**2.** (a) A conservative vector field $\mathbf{F}$ is a vector field which is the gradient of some scalar function $f$.

(b) The function $f$ in part (a) is called a potential function for $\mathbf{F}$, that is, $\mathbf{F} = \nabla f$.

**4.** (a) See Definition 17.2.13 [ET 16.2.13].

(b) If $\mathbf{F}$ is a force field, $\int_C \mathbf{F} \cdot d\mathbf{r}$ represents the work done by $\mathbf{F}$ in moving a particle along the curve $C$.

(c) $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz$

**6.** (a) $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path if the line integral has the same value for any two curves that have the same initial and terminal points.

(b) See Theorem 17.3.4 [ET 16.3.4].

**8.** See Equations 17.4.5 [ET 16.4.5].

**10.** See Theorem 17.3.6 [ET 16.3.6]; see Theorem 17.5.4 [ET 16.5.4].

**12.** (a) See (1) in Section 17.7 [ET 16.7].

(b) We normally evaluate the surface integral using Formula 17.7.2 [ET 16.7.2].

(c) See Formula 17.7.4 [ET 16.7.4].

(d) The mass is $m = \iint_S \rho(x, y, z)\,dS$ and the center of mass is $(\overline{x}, \overline{y}, \overline{z})$ where $\overline{x} = \frac{1}{m}\iint_S x\rho(x, y, z)\,dS$, $\overline{y} = \frac{1}{m}\iint_S y\rho(x, y, z)\,dS$, $\overline{z} = \frac{1}{m}\iint_S z\rho(x, y, z)\,dS$.

**14.** See the statement of Stokes' Theorem on page 1129 [ET 1093.].

**16.** In each theorem, we have an integral of a "derivative" over a region on the left side, while the right side involves the values of the original function only on the boundary of the region.

**1.** False; div $\mathbf{F}$ is a scalar field.

**2.** True. (See Definition 17.5.1 [ET 16.5.1].)

**3.** True, by Theorem 17.5.3 [ET 16.5.3] and the fact that div $\mathbf{0} = 0$.

**4.** True, by Theorem 17.3.2 [ET 16.3.2].

**5.** False. See Exercise 17.3.33 [ET 16.3.33]. (But the assertion is true if $D$ is simply-connected; see Theorem 17.3.6 [ET 16.3.6].)

**6.** False. See the discussion accompanying Figure 8 on page 1075 [ET 1039].

**7.** True. Apply the Divergence Theorem and use the fact that div $\mathbf{F} = 0$.

**8.** False by Theorem 17.5.11 [ET 16.5.11], because if it were true, then $\operatorname{div} \operatorname{curl} \mathbf{F} = 3 \neq 0$.

**2.** We can parametrize $C$ by $x = x$, $y = x^2$, $0 \le x \le 1$ so

$$\int_C x\,ds = \int_0^1 x\sqrt{1+(2x)^2}\,dx = \tfrac{1}{12}(1+4x^2)^{3/2}\Big]_0^1 = \tfrac{1}{12}\left(5\sqrt{5}-1\right).$$

**4.** $x = 3\cos t \;\Rightarrow\; dx = -3\sin t\,dt$, $y = 2\sin t \;\Rightarrow\; dy = 2\cos t\,dt$, $0 \le t \le 2\pi$, so

$$\begin{aligned}\int_C y\,dx + (x+y^2)\,dy &= \int_0^{2\pi}\left[(2\sin t)(-3\sin t) + (3\cos t + 4\sin^2 t)(2\cos t)\right]dt \\ &= \int_0^{2\pi}(-6\sin^2 t + 6\cos^2 t + 8\sin^2 t\,\cos t)\,dt = \int_0^{2\pi}\left[6(\cos^2 t - \sin^2 t) + 8\sin^2 t\,\cos t\right]dt \\ &= \int_0^{2\pi}(6\cos 2t + 8\sin^2 t\,\cos t)\,dt = 3\sin 2t + \tfrac{8}{3}\sin^3 t\Big]_0^{2\pi} = 0\end{aligned}$$

*Or*: Notice that $\frac{\partial}{\partial y}(y) = 1 = \frac{\partial}{\partial x}(x+y^2)$, so $\mathbf{F}(x,y) = \langle y, x+y^2\rangle$ is a conservative vector field. Since $C$ is a closed curve, $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C y\,dx + (x+y^2)\,dy = 0$.

**6.** $\int_C \sqrt{xy}\,dx + e^y\,dy + xz\,dz = \int_0^1 \left(\sqrt{t^4\cdot t^2}\cdot 4t^3 + e^{t^2}\cdot 2t + t^4\cdot t^3\cdot 3t^2\right)dt = \int_0^1 (4t^6 + 2te^{t^2} + 3t^9)\,dt$

$$= \left[\tfrac{4}{7}t^7 + e^{t^2} + \tfrac{3}{10}t^{10}\right]_0^1 = e - \tfrac{9}{70}$$

**8.** $\mathbf{F}(\mathbf{r}(t)) = (\sin t)(1+t)\,\mathbf{i} + (\sin^2 t)\,\mathbf{j}$, $\mathbf{r}'(t) = \cos t\,\mathbf{i} + \mathbf{j}$ and

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{\pi}\left((1+t)\sin t\cos t + \sin^2 t\right)dt = \int_0^{\pi}\left(\tfrac{1}{2}(1+t)\sin 2t + \sin^2 t\right)dt \\ &= \left[\tfrac{1}{2}\left((1+t)\left(-\tfrac{1}{2}\cos 2t\right) + \tfrac{1}{4}\sin 2t\right) + \tfrac{1}{2}t - \tfrac{1}{4}\sin 2t\right]_0^{\pi} = \tfrac{\pi}{4}\end{aligned}$$

**10.** (a) $C$: $x = 3-3t$, $y = \frac{\pi}{2}t$, $z = 3t$, $0 \le t \le 1$. Then

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \left[3t\,\mathbf{i} + (3-3t)\,\mathbf{j} + \tfrac{\pi}{2}t\,\mathbf{k}\right]\cdot\left[-3\,\mathbf{i} + \tfrac{\pi}{2}\,\mathbf{j} + 3\,\mathbf{k}\right]dt = \int_0^1 \left[-9t + \tfrac{3\pi}{2}\right]dt = \tfrac{1}{2}(3\pi - 9).$$

(b) $W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi/2}(3\sin t\,\mathbf{i} + 3\cos t\,\mathbf{j} + t\,\mathbf{k})\cdot(-3\sin t\,\mathbf{i} + \mathbf{j} + 3\cos t\,\mathbf{k})\,dt$

$$\begin{aligned}&= \int_0^{\pi/2}(-9\sin^2 t + 3\cos t + 3t\cos t)\,dt = \left[-\tfrac{9}{2}(t - \sin t\cos t) + 3\sin t + 3(t\sin t + \cos t)\right]_0^{\pi/2} \\ &= -\tfrac{9\pi}{4} + 3 + \tfrac{3\pi}{2} - 3 = -\tfrac{3\pi}{4}\end{aligned}$$

**12.** $\mathbf{F}$ is defined on all of $\mathbb{R}^3$, its components have continuous partial derivatives, and $\operatorname{curl}\mathbf{F} = (0-0)\,\mathbf{i} - (0-0)\,\mathbf{j} + (\cos y - \cos y)\,\mathbf{k} = \mathbf{0}$, so $\mathbf{F}$ is conservative by Theorem 17.5.4 [ET 16.5.4]. Thus there exists a function $f$ such that $\nabla f = \mathbf{F}$. Then $f_x(x,y,z) = \sin y$ implies $f(x,y,z) = x\sin y + g(y,z)$ and then $f_y(x,y,z) = x\cos y + g_y(y,z)$. But $f_y(x,y,z) = x\cos y$, so $g_y(y,z) = 0 \;\Rightarrow\; g(y,z) = h(z)$. Then $f(x,y,z) = x\sin y + h(z)$ implies $f_z(x,y,z) = h'(z)$. But $f_z(x,y,z) = -\sin z$, so $h(z) = \cos z + K$. Thus a potential function for $\mathbf{F}$ is $f(x,y,z) = x\sin y + \cos z + K$.

**14.** Here $\operatorname{curl}\mathbf{F} = \mathbf{0}$, the domain of $\mathbf{F}$ is $\mathbb{R}^3$, and the components of $\mathbf{F}$ have continuous partial derivatives, so $\mathbf{F}$ is conservative. Furthermore $f(x,y,z) = xe^y + ye^z$ is a potential function for $\mathbf{F}$. Then $\int_C \mathbf{F}\cdot d\mathbf{r} = f(4,0,3) - f(0,2,0) = 4 - 2 = 2$.

**16.** $\int_C \sqrt{1+x^3}\,dx + 2xy\,dy = \iint_D \left[\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}\left(\sqrt{1+x^3}\right)\right] dA = \int_0^1 \int_0^{3x} (2y-0)\,dy\,dx = \int_0^1 9x^2\,dx = 3x^3\Big]_0^1 = 3$

**18.** $\operatorname{curl}\mathbf{F} = (0 - e^{-y}\cos z)\,\mathbf{i} - (e^{-z}\cos x - 0)\,\mathbf{j} + (0 - e^{-x}\cos y)\,\mathbf{k} = -e^{-y}\cos z\,\mathbf{i} - e^{-z}\cos x\,\mathbf{j} - e^{-x}\cos y\,\mathbf{k}$,

$\operatorname{div}\mathbf{F} = -e^{-x}\sin y - e^{-y}\sin z - e^{-z}\sin x$

**20.** Let $\mathbf{F} = P_1\,\mathbf{i} + Q_1\,\mathbf{j} + R_1\,\mathbf{k}$ and $\mathbf{G} = P_2\,\mathbf{i} + Q_2\,\mathbf{j} + R_2\,\mathbf{k}$ be vector fields whose first partials exist and are continuous. Then

$$\begin{aligned}\mathbf{F}\,\mathrm{div}\,\mathbf{G} - \mathbf{G}\,\mathrm{div}\,\mathbf{F} = &\left[P_1\left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z}\right)\mathbf{i} + Q_1\left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z}\right)\mathbf{j} + R_1\left(\frac{\partial P_2}{\partial x} + \frac{\partial Q_2}{\partial y} + \frac{\partial R_2}{\partial z}\right)\mathbf{k}\right] \\ &- \left[P_2\left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right)\mathbf{i} + Q_2\left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right)\mathbf{j}\right. \\ &\left. + R_2\left(\frac{\partial P_1}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R_1}{\partial z}\right)\mathbf{k}\right]\end{aligned}$$

and

$$\begin{aligned}(\mathbf{G}\cdot\nabla)\,\mathbf{F} - (\mathbf{F}\cdot\nabla)\,\mathbf{G} = &\left[\left(P_2\frac{\partial P_1}{\partial x} + Q_2\frac{\partial P_1}{\partial y} + R_2\frac{\partial P_1}{\partial z}\right)\mathbf{i} + \left(P_2\frac{\partial Q_1}{\partial x} + Q_2\frac{\partial Q_1}{\partial y} + R_2\frac{\partial Q_1}{\partial z}\right)\mathbf{j}\right. \\ &\left. + \left(P_2\frac{\partial R_1}{\partial x} + Q_2\frac{\partial R_1}{\partial y} + R_2\frac{\partial R_1}{\partial z}\right)\mathbf{k}\right] \\ &- \left[\left(P_1\frac{\partial P_2}{\partial x} + Q_1\frac{\partial P_2}{\partial y} + R_1\frac{\partial P_2}{\partial z}\right)\mathbf{i} + \left(P_1\frac{\partial Q_2}{\partial x} + Q_1\frac{\partial Q_2}{\partial y} + R_1\frac{\partial Q_2}{\partial z}\right)\mathbf{j}\right. \\ &\left. + \left(P_1\frac{\partial R_2}{\partial x} + Q_1\frac{\partial R_2}{\partial y} + R_1\frac{\partial R_2}{\partial z}\right)\mathbf{k}\right]\end{aligned}$$

Hence

$$\begin{aligned}&\mathbf{F}\,\mathrm{div}\,\mathbf{G} - \mathbf{G}\,\mathrm{div}\,\mathbf{F} + (\mathbf{G}\cdot\nabla)\,\mathbf{F} - (\mathbf{F}\cdot\nabla)\,\mathbf{G} \\ &= \left[\left(P_1\frac{\partial Q_2}{\partial y} + Q_2\frac{\partial P_1}{\partial x}\right) - \left(P_2\frac{\partial Q_1}{\partial y} + Q_1\frac{\partial P_2}{\partial y}\right)\right. \\ &\qquad \left. - \left(P_2\frac{\partial R_1}{\partial z} + R_1\frac{\partial P_2}{\partial z}\right) + \left(P_1\frac{\partial R_2}{\partial z} + R_2\frac{\partial P_1}{\partial z}\right)\right]\mathbf{i} \\ &\quad + \left[\left(Q_1\frac{\partial R_2}{\partial z} + R_2\frac{\partial Q_1}{\partial z}\right) - \left(Q_2\frac{\partial R_1}{\partial z} + R_1\frac{\partial Q_2}{\partial z}\right)\right. \\ &\qquad \left. - \left(P_1\frac{\partial Q_2}{\partial x} + Q_2\frac{\partial P_1}{\partial x}\right) + \left(P_2\frac{\partial Q_1}{\partial x} + Q_1\frac{\partial P_2}{\partial x}\right)\right]\mathbf{j} \\ &\quad + \left[\left(P_2\frac{\partial R_1}{\partial x} + R_1\frac{\partial P_2}{\partial x}\right) - \left(P_1\frac{\partial R_2}{\partial x} + R_2\frac{\partial P_1}{\partial x}\right)\right. \\ &\qquad \left. - \left(Q_1\frac{\partial R_2}{\partial y} + R_2\frac{\partial Q_1}{\partial y}\right) + \left(Q_2\frac{\partial R_1}{\partial y} + R_1\frac{\partial Q_2}{\partial y}\right)\right]\mathbf{k} \\ &= \left[\frac{\partial}{\partial y}(P_1Q_2 - P_2Q_1) - \frac{\partial}{\partial z}(P_2R_1 - P_1R_2)\right]\mathbf{i} \\ &\quad + \left[\frac{\partial}{\partial z}(Q_1R_2 - Q_2R_1) - \frac{\partial}{\partial x}(P_1Q_2 - P_2Q_1)\right]\mathbf{j} \\ &\quad + \left[\frac{\partial}{\partial x}(P_2R_1 - P_1R_2) - \frac{\partial}{\partial y}(Q_1R_2 - Q_2R_1)\right]\mathbf{k} \\ &= \mathrm{curl}\,(\mathbf{F}\times\mathbf{G})\end{aligned}$$

**22.** $\nabla^2(fg) = \dfrac{\partial^2(fg)}{\partial x^2} + \dfrac{\partial^2(fg)}{\partial y^2} + \dfrac{\partial^2(fg)}{\partial z^2}$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\,g + f\,\frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\,g + f\,\frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\,g + f\,\frac{\partial g}{\partial z}\right) \quad \text{[Product Rule]}$$

$$= \frac{\partial^2 f}{\partial x^2}\,g + 2\,\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + f\,\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\,g + 2\,\frac{\partial f}{\partial y}\frac{\partial g}{\partial y}$$

$$+ f\,\frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\,g + 2\,\frac{\partial f}{\partial z}\frac{\partial g}{\partial z} + f\,\frac{\partial^2 g}{\partial z^2} \quad \text{[Product Rule]}$$

$$= f\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}\right) + g\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) + 2\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right\rangle \cdot \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right\rangle$$

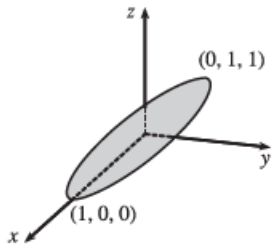
$$= f\nabla^2 g + g\nabla^2 f + 2\nabla f \cdot \nabla g$$

*Another method:* Using the rules in Exercises 15.6.37(b) [ET 14.6.37(b)] and 17.5.25 [ET 16.5.25], we have

$$\nabla^2(fg) = \nabla \cdot \nabla(fg) = \nabla \cdot (g\,\nabla f + f\,\nabla g) = \nabla g \cdot \nabla f + g\nabla \cdot \nabla f + \nabla f \cdot \nabla g + f\nabla \cdot \nabla g$$

$$= g\,\nabla^2 f + f\,\nabla^2 g + 2\nabla f \cdot \nabla g$$

**24.** (a) $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, so $C$ lies on the circular cylinder $x^2 + y^2 = 1$. But also $y = z$, so $C$ lies on the plane $y = z$. Thus $C$ is the intersection of the plane $y = z$ and the cylinder $x^2 + y^2 = 1$.



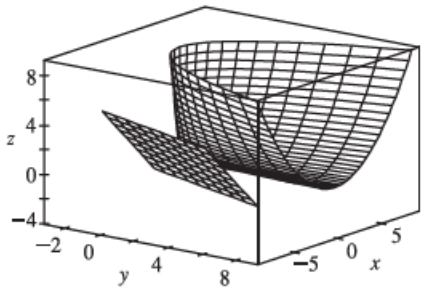

(b) Apply Stokes' Theorem, $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S}$:

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xe^{2y} & 2x^2e^{2y} + 2y\cot z & -y^2\csc^2 z \end{vmatrix} = \left\langle -2y\csc^2 z - (-2y\csc^2 z),\, 0,\, 4xe^{2y} - 4xe^{2y}\right\rangle = \mathbf{0}$$

Therefore $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \mathbf{0} \cdot d\mathbf{S} = 0$.

**26.** (a) $\mathbf{r}_u = -v\,\mathbf{j} + 2u\,\mathbf{k}$, $\mathbf{r}_v = 2v\,\mathbf{i} - u\,\mathbf{j}$ and $\mathbf{r}_u \times \mathbf{r}_v = 2u^2\,\mathbf{i} + 4uv\,\mathbf{j} + 2v^2\,\mathbf{k}$. Since the point $(4, -2, 1)$ corresponds to $u = 1$, $v = 2$ (or $u = -1$, $v = -2$ but $\mathbf{r}_u \times \mathbf{r}_v$ is the same for both), a normal vector to the surface at $(4, -2, 1)$ is $2\,\mathbf{i} + 8\,\mathbf{j} + 8\,\mathbf{k}$ and an equation of the tangent plane is $2x + 8y + 8z = 0$ or $x + 4y + 4z = 0$.

(b)




(c) By Definition 17.6.6 [ET 16.6.6], the area of $S$ is given by

$$A(S) = \int_0^3 \int_{-3}^3 \sqrt{(2u^2)^2 + (4uv)^2 + (2v^2)^2}\,dv\,du = 2\int_0^3 \int_{-3}^3 \sqrt{u^4 + 4u^2v^2 + v^4}\,dv\,du.$$

(d) By Equation 17.7.9 [ET 16.7.9], the surface integral is

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \int_0^3 \int_{-3}^3 \left\langle \frac{(u^2)^2}{1+(v^2)^2}, \frac{(v^2)^2}{1+(-uv)^2}, \frac{(-uv)^2}{1+(u^2)^2} \right\rangle \cdot \left\langle 2u^2, 4uv, 2v^2 \right\rangle dv\,du \\ &= \int_0^3 \int_{-3}^3 \left( \frac{2u^6}{1+v^4} + \frac{4uv^5}{1+u^2v^2} + \frac{2u^2v^4}{1+u^4} \right) dv\,du \approx 1524.0190\end{aligned}$$

**28.** $z = f(x, y) = 4 + x + y$ with $0 \le x^2 + y^2 \le 4$ so $\mathbf{r}_x \times \mathbf{r}_y = -\mathbf{i} - \mathbf{j} + \mathbf{k}$. Then

$$\begin{aligned}\iint_S (x^2 z + y^2 z)\,dS &= \iint_{x^2+y^2\le 4} (x^2 + y^2)(4 + x + y)\sqrt{3}\,dA \\ &= \int_0^2 \int_0^{2\pi} \sqrt{3}\,r^3(4 + r\cos\theta + r\sin\theta)\,d\theta\,dr = \int_0^2 8\pi\sqrt{3}\,r^3\,dr = 32\pi\sqrt{3}\end{aligned}$$

**30.** $z = f(x, y) = x^2 + y^2$, $\mathbf{r}_x \times \mathbf{r}_y = -2x\,\mathbf{i} - 2y\,\mathbf{j} + \mathbf{k}$ (because of upward orientation) and $\mathbf{F}(\mathbf{r}(x, y)) \cdot (\mathbf{r}_x \times \mathbf{r}_y) = -2x^3 - 2xy^2 + x^2 + y^2$. Then

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_{x^2+y^2\le 1} (-2x^3 - 2xy^2 + x^2 + y^2)\,dA \\ &= \int_0^1 \int_0^{2\pi} (-2r^3\cos^3\theta - 2r^3\cos\theta\,\sin^2\theta + r^2)\,r\,dr\,d\theta = \int_0^1 r^3(2\pi)\,dr = \tfrac{\pi}{2}\end{aligned}$$

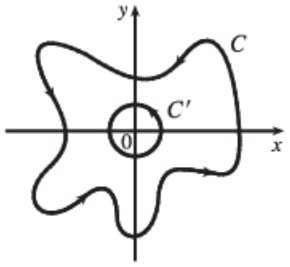
**32.** $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \oint_C \mathbf{F}\cdot d\mathbf{r}$ where $C$: $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + \mathbf{k}$, $0 \le t \le 2\pi$, so $\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$, $\mathbf{F}(\mathbf{r}(t)) = 8\cos^2 t\,\sin t\,\mathbf{i} + 2\sin t\,\mathbf{j} + e^{4\cos t\sin t}\,\mathbf{k}$, and $\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) = -16\cos^2 t\sin^2 t + 4\sin t\cos t$. Thus

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} (-16\cos^2 t\,\sin^2 t + 4\sin t\,\cos t)\,dt = \left[-16\left(-\tfrac{1}{4}\sin t\,\cos^3 t + \tfrac{1}{16}\sin 2t + \tfrac{1}{8}t\right) + 2\sin^2 t\right]_0^{2\pi} = -4\pi.$$

**34.** $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E 3(x^2 + y^2 + z^2)\,dV = \int_0^{2\pi}\int_0^1\int_0^2 (3r^2 + 3z^2)\,r\,dz\,dr\,d\theta = 2\pi\int_0^1 (6r^3 + 8r)\,dr = 11\pi$

**36.** Here we must use Equation 17.9.7 [ET 16.9.7] since $\mathbf{F}$ is not defined at the origin. Let $S_1$ be the sphere of radius 1 with center at the origin and outer unit normal $\mathbf{n}_1$. Let $S_2$ be the surface of the ellipsoid with outer unit normal $\mathbf{n}_2$ and let $E$ be the solid region between $S_1$ and $S_2$. Then the outward flux of $\mathbf{F}$ through the ellipsoid is given by $\iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2 \, dS = -\iint_{S_1} \mathbf{F} \cdot (-\mathbf{n}_1) \, dS + \iiint_E \operatorname{div} \mathbf{F} \, dV$. But $\mathbf{F} = \mathbf{r}/|\mathbf{r}|^3$, so $\operatorname{div} \mathbf{F} = \nabla \cdot \left(|\mathbf{r}|^{-3}\mathbf{r}\right) = |\mathbf{r}|^{-3}(\nabla \cdot \mathbf{r}) + \mathbf{r} \cdot \left(\nabla |\mathbf{r}|^{-3}\right) = |\mathbf{r}|^{-3}(3) + \mathbf{r} \cdot \left(-3|\mathbf{r}|^{-4}\right)\left(\mathbf{r}|\mathbf{r}|^{-1}\right) = 0$. [Here we have used Exercises 17.5.30(a) [ET 16.5.30(a)] and 17.5.31(a) [ET 16.5.31(a)].] And $\mathbf{F} \cdot \mathbf{n}_1 = \dfrac{\mathbf{r}}{|\mathbf{r}|^3} \cdot \dfrac{\mathbf{r}}{|\mathbf{r}|} = |\mathbf{r}|^{-2} = 1$ on $S_1$. Thus $\iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2 \, dS = \iint_{S_1} dS + \iiint_E 0 \, dV = (\text{surface area of the unit sphere}) = 4\pi(1)^2 = 4\pi$.

**38.** Let $C'$ be the circle with center at the origin and radius $a$ as in the figure. Let $D$ be the region bounded by $C$ and $C'$. Then $D$'s positively oriented boundary is $C \cup (-C')$. Hence by Green's Theorem




$$\int_C \mathbf{F} \cdot d\mathbf{r} + \int_{-C'} \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \mathbf{0}, \text{ so}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_{-C'} \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot dr = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt$$

$$= \int_0^{2\pi} \left[\frac{2a^3\cos^3 t + 2a^3 \cos t \, \sin^2 t - 2a \sin t}{a^2}(-a \sin t) + \frac{2a^3 \sin^3 t + 2a^3 \cos^2 t \, \sin t + 2a \cos t}{a^2}(a \cos t)\right] dt$$

$$= \int_0^{2\pi} \frac{2a^2}{a^2} \, dt = 4\pi$$

**40.** The stated conditions allow us to use the Divergence Theorem. Hence $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div}(\operatorname{curl} \mathbf{F}) \, dV = 0$ since $\operatorname{div}(\operatorname{curl} \mathbf{F}) = 0$.