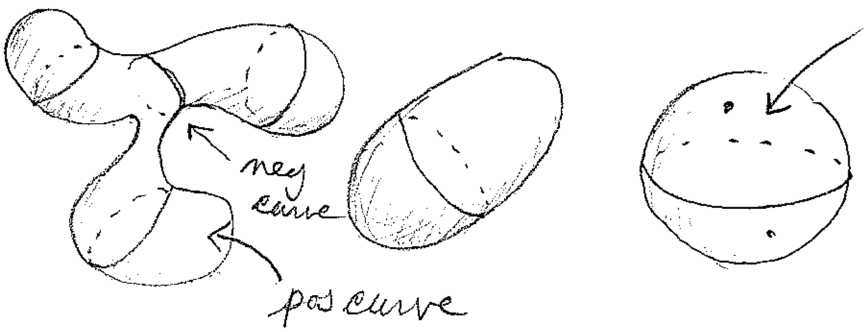
Lecture 40: Geometry, Topology and Number Theory.

Reminder: Final Thursday, May 14
8–11am. Comprehensive. 2 page "cheat sheet."

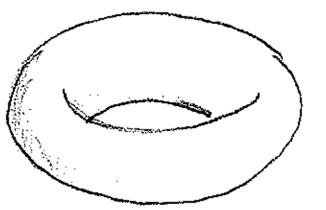
Geometry (study of metric spaces)



homogeneous positive curvature

Topologically the same, metrically different.

$$\mathrm{len}(C_r) = 2\pi r - \frac{K\pi}{3} r^3 + O(r^4)$$

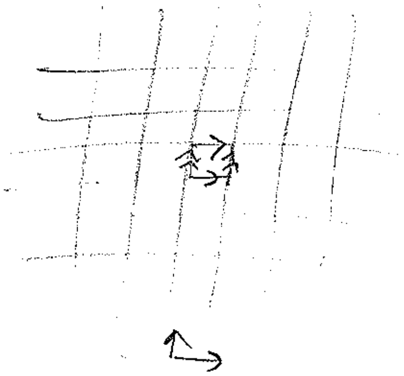


$\subseteq \mathbb{R}^3$

not homogeneous

$T = \mathbb{R}^2 / \mathbb{Z}^2$

homogeneous, flat.



[Q: What are some other compact surfaces?]

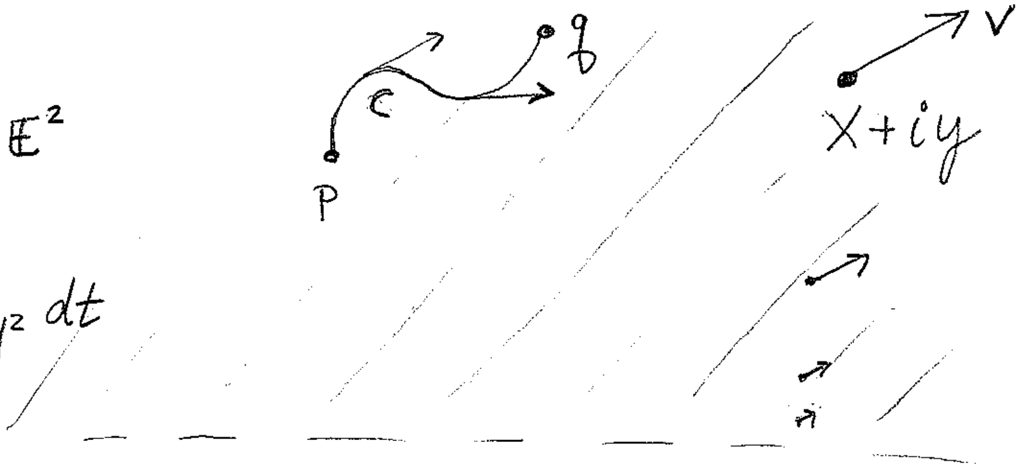
Q How could we construct such a thing?

(locally) homogeneous, constant neg. curvature

A. Quaternion algebras!

Hyperbolic Plane: $\mathbb{H}^2 = \{ z \in \mathbb{C} \mid \operatorname{Im}(z) > 0 \}$

$$\| v \|_{\mathbb{H}^2} = \frac{1}{y} \| v \|_{\mathbb{E}^2}$$



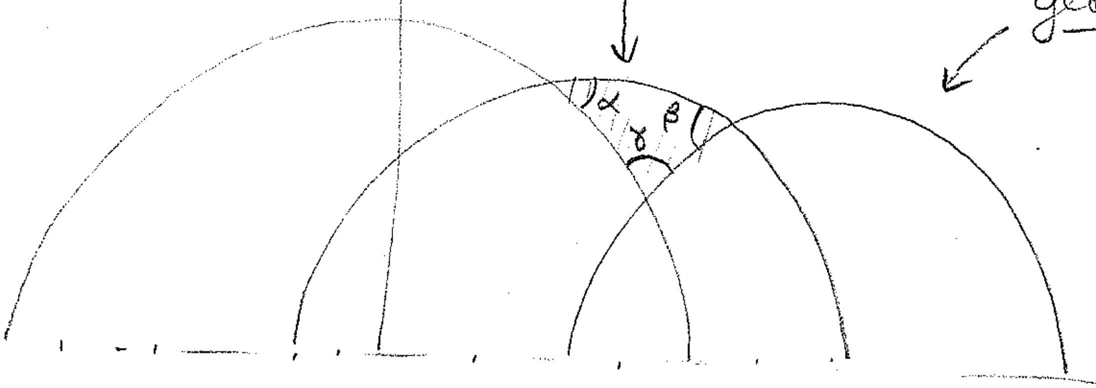
$$\operatorname{Len}(\alpha) = \int_a^b \| c'(t) \|_{\mathbb{H}^2} \, dt$$

$d(p,q) = \inf \{ \operatorname{Len}(c) \mid c \text{ a path from } p \text{ to } q \}$ ← complete metric

Compare:

area $= \pi - \alpha - \beta - \gamma$

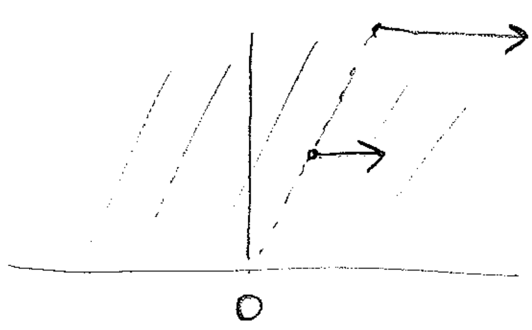
geodesics



Homogeneous:

Some isometries: $z \mapsto z + t$ for $t \in \mathbb{R}$

$z \mapsto az$ for $a \in \mathbb{R}_{>0}$.



All orient. pres. isometries

$$PSL_2\mathbb{R} \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ corresponds to}$$

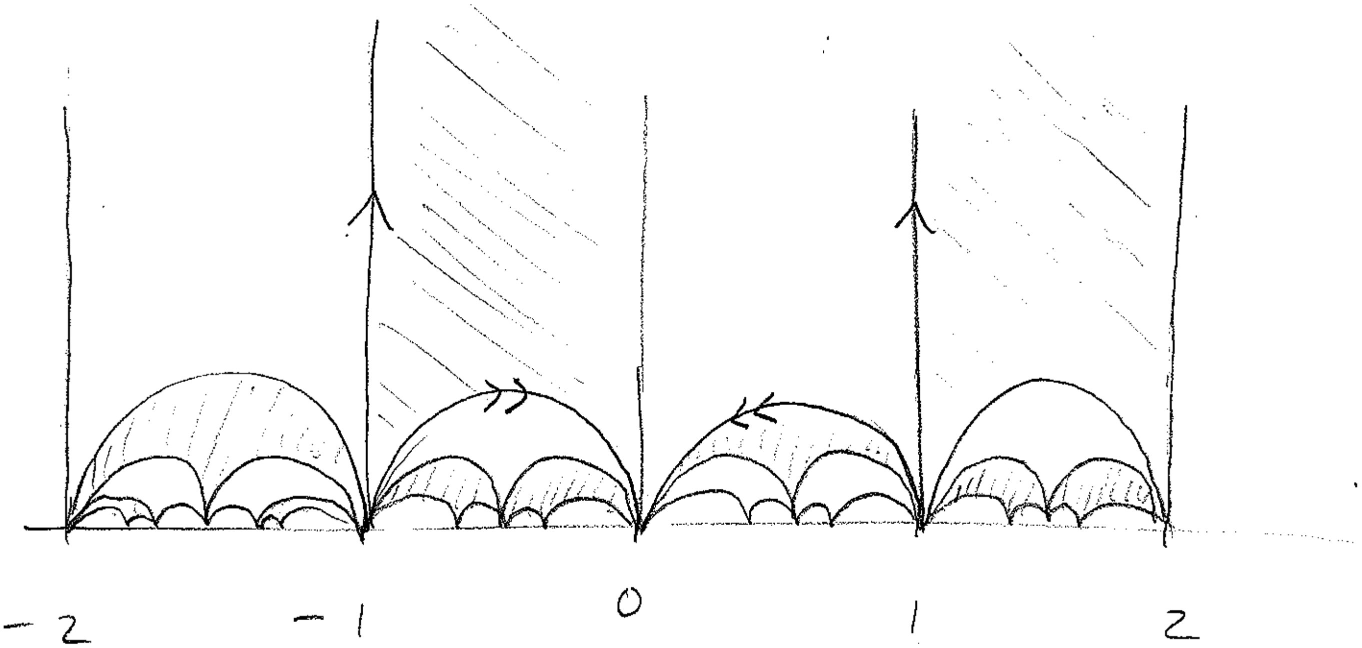
$$\| \qquad\qquad z \longmapsto \frac{az+b}{cz+d}$$

$$SL_2\mathbb{R}/_{\pm I}$$

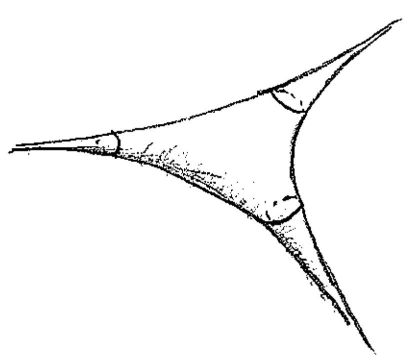
[Exactly the conformal automorphisms...]

$$\Gamma(2) = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle \subseteq PSL_2\mathbb{Z}$$

$\hookleftarrow$ index $= \# \, PSL_2\mathbb{F}_2 = 6$



$$z \longmapsto z+2 \qquad\qquad z \longmapsto \frac{z}{2z+1}$$

$$\mathbb{H}^2/\Gamma(2) =$$  $$= S^2 \setminus \{3\text{ pts}\}$$

$$\text{Area} = 2\pi.$$

$\left[\mathbb{H}^2/\Gamma(2)\right.$ = moduli space of elliptic curves, with a subgroup of order 4
Modular forms are cohomology classes$\left.\right]$

A a quaternion algebra / $\mathbb{Q}$ s.t.

① A is a division alg. (not $M_2(\mathbb{Q})$)

② $A_\infty \cong M_2(\mathbb{R})$ not $\mathbb{H}$.

↑ means we change the underlying field to $\mathbb{Q}_\infty = \mathbb{R}$

Now pick an <u>order</u> $\mathcal{O} \subseteq A$, i.e.
a subring with 1 where $(\mathcal{O}, +)$ is a complete lattice in $A \cong \mathbb{Q}^4$!

<u>Ex</u>: $A = \left(\frac{2,3}{\mathbb{Q}}\right)$

$$\mathcal{O} = \langle 1, \tfrac{1}{2}(1+j+k), \tfrac{1}{2}(1+j-k), \tfrac{1}{2}(i+k)\rangle$$

$$\mathcal{O} = \langle 1, i, j, k\rangle$$

Now look at

$$\mathcal{O}^1 \subseteq A^1 = \{\alpha \in A \mid \mathcal{N}(\alpha) = 1\}.$$

Now we have

$$G' \longrightarrow A^1_\infty = SL_2\mathbb{R}$$

no parabolics as no zero divisors

image Γ is discrete and $\mathbb{H}^2/\Gamma$ is compact.

compare Minkowski picture

[Up to finite index] Γ preserves the tiling on the handout.

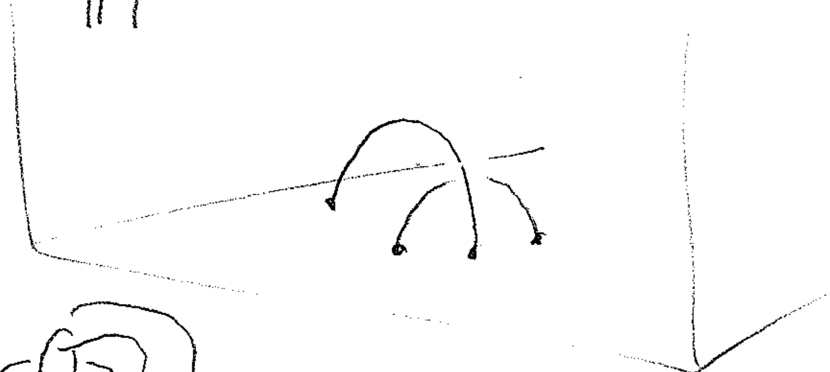
Thurston/Perelman; "Most" 3-manifolds are hyperbolic.

Special arithmetic constructed from quaternion algebras

$\mathbb{H}^3/PSL_2\mathbb{Z}[\zeta_3]$

$= S^3 \setminus$ 

$\mathbb{H}^3$



$\mathbb{H}^3/PSL_2\mathbb{Z}[i] = S^3 \setminus$ 

Automorphic forms can be used to control the (co)homology of such 3-manifolds with gon consequences about e.g. the surfaces inside them.

See

M. Emerton "3-manifolds and the Langlands program".