Lecture 39: Hasse Minkowski fin; Adèles + Idèles.

Hasse-Minkowski: A quad form over $\mathbb{Q}$ reps 0 iff it does so at each place $\mathbb{Q}_v$.

Pf: $n \geq 5$: Induct on n. Can take

$$q = \underbrace{a_1 X_1^2 + a_2 X_2^2}_{f} - \underbrace{(a_3 X_3^2 + \cdots + a_n X_n^n)}_{g} \qquad a_i \in \mathbb{Z}^\times$$

Idea: reduce to $bZ^2 - g$ where f reps $a \in \mathbb{Q}_s^\times$ and $q' = bZ^2 - g$ reps 0 at every v.

[Point q' rep's 0 $\Rightarrow$ q reps 0] Let $S = \{2, \infty\} \cup \{\text{primes div } a_i\}$.

As q_v reps 0, $\exists\, b_v \in \mathbb{Q}_v^\times$ with $f(x_1^v, x_2^v) = b_v = g(x_3^v, \dots, x_n^v)$

Set $f_v^{-1}(b_v (\mathbb{Q}_v)^2) = U_v \subseteq \mathbb{Q}_v \times \mathbb{Q}_v$, which is open by cont of f_v. By approx, $\exists (x_1, x_2) \in \mathbb{Q}^2$ lying in $\prod_{v \in S} U_v \subseteq \left(\prod_{v \in S} \mathbb{Q}_v\right)^2$. Then $b = f(x_1, x_2)$ has the prop that $b = b_v c_v^2$ in $\mathbb{Q}_v$ for $v \in S$.

Thus, for v in S, g_v reps b and so q_v' reps 0.

For $v \notin S$, we saw last time that g_v rep b and hence q_v' reps 0. By induction, q' reps 0 over $\mathbb{Q}$

$\Rightarrow q$ reps 0 over $\mathbb{Q}$. Q.E.D.

Remarks: ① Simple to state yet the proof uses many things, such as quad recip, Dirichlet's Thm...

② Encodes many classical results, e.g. every number is a sum of 4 squares. [Basically everything red to what happens mod 8...]

③ Key to understanding quaternion algebras.

Adèles and Idèles: $\mathcal{V}$ = places of $\mathbb{Q}$

Consider $\mathbb{Q} \longrightarrow \prod_{v \in \mathcal{V}} \mathbb{Q}_v = \mathbb{R} \times \mathbb{Q}_2 \times \mathbb{Q}_3 \times \mathbb{Q}_5 \times \cdots$

$x \longmapsto (x, x, x, \dots)$

Image lies in this subring, called the adèles

$$\mathbb{A} = \left\{ (x_v) \in \prod_v \mathbb{Q}_v \;\middle|\; x_v \in \mathbb{Z}_v \text{ for almost all } v \right\}$$

Topologize as follows (not the prod. top!)

A basis about 0 consists of sets of the form $U = \prod_v U_v$ where $U_v \subseteq \mathbb{Q}_v$ is an open set containing 0 and almost all $U_v = \mathbb{Z}_v$

More open sets than the prod. top, e.g.

$U = (-1,1)\times\prod_{v\neq\infty}\mathbb{Z}_v$ is open. However, the top restricted to U or $\bar{U} = [-1,1]\times\prod_{v\neq\infty}\mathbb{Z}_p$ is just the prod. top. Note $\bar{U}$ is compact, and in fact $\mathbb{A}_K$ is a locally compact topological ring

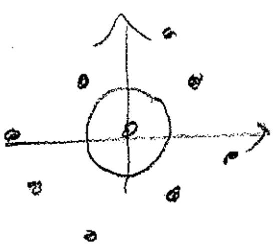
<u>Prop</u>: $\mathbb{Q}$ is discrete in $\mathbb{A}$, and $\mathbb{A}/\mathbb{Q}$ is compact.

<u>Compare</u>: $\mathbb{Z}\subseteq\mathbb{R}$ is discrete, $\mathbb{R}/\mathbb{Z} = \bigcirc = S^1$

$\mathcal{O}_K \to K_{\mathbb{R}}$ is discrete with $K_{\mathbb{R}}/\mathcal{O}_K = (S^1)^n$

[In general, any number field has a ring of adèles]

<u>Proof of discreteness</u>: Enough to show $\exists\, U^{\text{open}} \subseteq \mathbb{A}$ with $U\cap\mathbb{Q} = \{0\}$. Can take $U = (-1/2, 1/2)\times\prod_{v\neq\infty}\mathbb{Z}_v$,



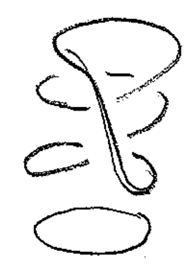
since if $x\in\mathbb{Q}^\times$, then $x\in\mathbb{Z}_v$ for all v $\Rightarrow x\in\mathbb{Z}$.

<u>Compactness</u>: $\bar{U} \twoheadrightarrow \mathbb{A}/\mathbb{Q}$.

<u>Idea</u>: Given $a\in\mathbb{A}$, trans by an elt of $\mathbb{Q}$ so that its in $\mathbb{R}\times\prod\mathbb{Z}_v$, then trans by $\lfloor x_\infty\rfloor\in\mathbb{Z}$. $\blacksquare$

Topologically, $\mathbb{A}/\mathbb{Q} = \varprojlim \mathbb{R}/n\mathbb{Z}$

a <u>solenoid</u> (Cantor set bundle over S^1!)



$z \mapsto z^3$

<u>Note</u>: $\mathbb{A}/\mathbb{Q}$ is the <u>Pontryagin dual</u> of $(\mathbb{Q},+)$ needed to do Fourier analysis on $\mathbb{Q}$.
(cf. $S^1 \leftrightarrow \mathbb{Z}$ and $\mathbb{R} \leftrightarrow \mathbb{R}$).

<u>Idèles</u>: $\mathbb{I}_K = \mathbb{A}_K^\times$ (with a slightly diff top)

← some # field

$K^\times$ is discrete in $\mathbb{I}_K$, so consider the idèle class group $C_K = \mathbb{I}_K / K^\times$. This isn't compact, but we set $\mathbb{I}_\infty = \prod_{v_\infty} K_{v_\infty}^\times \times \prod_{v_f} \mathcal{O}_{K_v}^\times$
then

$$Cl_K = \mathbb{I}_K / K^\times \cdot \mathbb{I}_\infty$$

usual ideal class group