Lecture 24: Units in integer rings

Goal:

Dirichlet's Unit Theorem: $\mathcal{O}_K^\times \cong \mu(K) \oplus \mathbb{Z}^{r+s-1}$

where

$$\mu(K) = \{\alpha \in \mathcal{O}_K^\times \mid \alpha \text{ has finite order}\} = \left\{\begin{matrix}\text{roots of}\\ \text{unity in } K\end{matrix}\right\}$$

$r = \#$ of real emb. $s = \#$ of pairs of comp emb.

Setup:

$$(z_\tau) \longmapsto (\log|z_\tau|)$$

$$\begin{tikzcd}
K^\times \arrow[r, "j"] \arrow[d, "N_{K/\mathbb{Q}}"'] \arrow[dr, phantom, "\circlearrowleft"] & K_{\mathbb{C}}^\times := \prod_\tau \mathbb{C}^* \arrow[r, "\ell"] \arrow[d, "N = \text{prod of coor}"] \arrow[dr, phantom, "\circlearrowleft"] & \prod_\tau \mathbb{R} \arrow[d, "Tr = \text{sum coor}"] \\
\mathbb{Q}^\times \arrow[r] & \mathbb{C}^\times \arrow[r, "\ell"] & \mathbb{R}
\end{tikzcd}$$

Now the generator F of $Gal(\mathbb{C}/\mathbb{R})$ acts on this diagram; taking fixed points yields

$$\begin{tikzcd}
K^\times \arrow[r] \arrow[d] & K_{\mathbb{R}}^\times \arrow[r] \arrow[d] & \left[\prod_\tau \mathbb{R}\right]_F \cong \mathbb{R}^{r+s} \arrow[d, "tr"] \\
\mathbb{Q}^\times \arrow[r] & \mathbb{R}^\times \arrow[r] & \mathbb{R}
\end{tikzcd}$$

Have

$$\mathcal{O}_K^\times = \{\alpha \in \mathcal{O}_K \mid N_{K/\mathbb{Q}}(\alpha) = \pm 1\}$$

and set

$$S = \{\vec{x} \in K_{\mathbb{R}}^\times \mid N(\vec{x}) = \pm 1\} \text{ and}$$

$$H = \{ \vec{y} \in [\prod_{\tau} \mathbb{R}]_F \mid tr(\vec{y}) = 0 \}$$

Have

$$\mathcal{O}_K^\times \xrightarrow{j} S \xrightarrow{l} H$$

and set $\Gamma = l(j(\mathcal{O}_K^\times))$ an additive subgroup of $H \cong \mathbb{R}^{r+s-1}$.

<u>Lemma 1</u>: $\Gamma \cong \mathcal{O}_K^\times / \mu(K)$.

<u>Lemma 2</u>: Γ is a complete lattice in H. $\Rightarrow \Gamma \cong \mathbb{Z}^{r+s-1}$.

Together, these will give the Thm since for any finitely gen abelian gp A we have

$$A \cong (A/_T) \oplus T \text{ where } T = \{ a \in A \mid a \text{ has finite order} \}$$

[follows e.g. from the classification of A.]

<u>Pf of Lemma 1</u>: Set $\lambda = l \circ j : \mathcal{O}_K^\times \to \Gamma$.
Must show $\ker \lambda = \mu(K)$. If $\zeta \in \mu(K)$, then $|\tau(\zeta)| = 1$ for all τ, and so $\lambda(\zeta) = \vec{0}$.
Conversely, suppose $\varepsilon \in \ker(\lambda)$. Then $|\tau(\varepsilon)| = 1$

for all τ. Thus $j(\varepsilon)$ lies a bounded

part of $j(\mathcal{O}_K^\times) \subseteq K_{\mathbb{R}}^*$ $\Longrightarrow$ $j(\ker\lambda)$ is finite

$\quad\quad\quad\quad \cap|$

$\quad\quad\quad\quad j(\mathcal{O}_K) \subseteq K_{\mathbb{R}}$ $\Longrightarrow$ $\ker\lambda$ is finite

$\quad\quad\quad\quad\uparrow$ complete lattice! $\Longrightarrow$ ε has finite order, i.e. is in $\mu(K)$. $\blacksquare$

Γ a complete lattice is equiv to

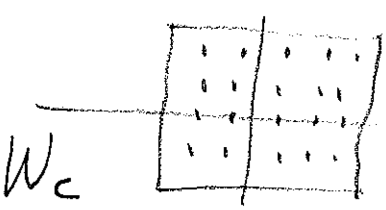
a) Γ is a discrete subset of H, i.e. $\forall h \in H, \exists \varepsilon > 0$ s.t. $B_\varepsilon(h) \cap H = \{h\}$

b) H/Γ is compact.

Pf of Lemma 2:

Discreteness: Enough to show that $\forall c > 0$, $U_c = \{(r_\tau) \in H \mid \|r\| < c\}$ contains finitely many pts of Γ. Now $\ell^{-1}(U_c)$

$\subseteq \{(z_\tau) \in K_{\mathbb{R}}^\times \mid e^{-c} < |z_\tau| < e^c\} = W_c$



Now $j(\mathcal{O}_K)$ is a lattice in $K_{\mathbb{C}}$ and so there are only finitely many such elts in W_c. Thus U_c contains only finitely many points of Γ. As a diagram

$$\begin{tikzcd}
K^\times \arrow[r, "j"] & \prod_\tau \mathbb{C}^\times \arrow[r, "\ell"] & \prod_\tau \mathbb{R} \\
\mathcal{O}_K^\times \arrow[r] & j(\mathcal{O}_K^\times) \arrow[r] & \Gamma \\
 & W_c \arrow[r] & U_c
\end{tikzcd}$$

<u>Compactness</u> <u>of</u> <u>H/Γ</u>: