Lecture 23: Units of integer rings

K a number field. Consider the units of $\mathcal{O}_K$:

$$\mathcal{O}_K^\times = \{\alpha \in \mathcal{O}_K \mid \exists \beta \in \mathcal{O}_K \text{ with } \alpha\beta = 1\} = \{\alpha \in \mathcal{O}_K \mid \alpha^{-1} \in \mathcal{O}_K\}$$
$$= \{\alpha \in \mathcal{O}_K \mid \underbrace{N_{K/\mathbb{Q}}(\alpha)}_{\alpha\cdot\beta \text{ for some } \beta \in \mathcal{O}_K} = \pm 1\}$$

Exs: ① $K = \mathbb{Q}(i)$, $\mathbb{Z}[i]^\times = \{1, -1, i, -i\}$

② $K = \mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\zeta_3)$, then $\mathcal{O}_K^\times = \{1, -1, \zeta_3, \bar{\zeta}_3, \zeta_6, \bar{\zeta}_6\}$, where $\frac{1}{2} - \frac{\sqrt{3}}{2}i = \zeta_3$ and $\frac{1}{2} + \frac{\sqrt{3}}{2}i = \zeta_6$

[In both these cases, the units have a geom. interpretation: they're the lattice pts of $\mathcal{O}_K$ on the unit circle.]

③ $K = \mathbb{Q}(\sqrt{2})$, $\mathcal{O}_K = \mathbb{Z}[\sqrt{2}]$. Then $\mathcal{O}_K^\times \cong \underset{\langle -1\rangle}{\mathbb{Z}/2\mathbb{Z}} \oplus \underset{\langle 1+\sqrt{2}\rangle}{\mathbb{Z}}$
which is plausible since $(1+\sqrt{2})(-1+\sqrt{2}) = 1$.

Connection to classical Diophantine equations: $x + \sqrt{2}y$ for $x, y \in \mathbb{Z}$ is a unit iff $x^2 - 2y^2 = \pm 1$.

—— ∘ ——

Consider
$$\mu(K) = \left\{\alpha \in \mathcal{O}_K^\times \;\middle|\; \begin{matrix}\alpha \text{ has finite order,}\\ \text{i.e. } \alpha^m = 1 \text{ for some } m\end{matrix}\right\} = \left\{\begin{matrix}\text{roots of}\\ \text{unity in } K\end{matrix}\right\}$$
which is finite as there finitely many $\mathbb{Q}(\zeta_m)$ of bounded degree. More precisely, $\mu(K) \cong \mathbb{Z}/M\mathbb{Z}$ where M is the largest m s.t. $\mathbb{Q}(\zeta_m) \subseteq K$.

Dirichlet's Unit Theorem: $\mathcal{O}_K^\times \cong \mu(K) \oplus \mathbb{Z}^{r+s-1}$

[where r = # of real emb,
s = # of pairs of comp emb.]

Tool: multiplicative Minkowski:

Consider:

$$
\begin{array}{ccccc}
k & \longmapsto & (\tau(k)) \quad (z_\tau) & \longmapsto & \prod z_\tau \\
K^\times & \xrightarrow{j} & K_{\mathbb{C}}^\times := \prod_\tau \mathbb{C}^\times & \xrightarrow{N} & \mathbb{C}^\times
\end{array}
$$

Note: $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$

To get a lattice out of $\mathcal{O}^\times$, we'll take logs as follows.

Define $l: \mathbb{C}^\times \to \mathbb{R}^\times$ by $l(z) = \log|z|$

which induces $l: K_{\mathbb{C}}^* \to \prod_\tau \mathbb{R}$ coordinate-wise.

We have the commutative diagram

$$
\begin{tikzcd}
K^\times \arrow[r, "j"] \arrow[d, "N_{K/\mathbb{Q}}"'] \arrow[dr, phantom, "\circlearrowright"] & K_{\mathbb{C}}^\times \arrow[r, "l"] \arrow[d, "N"] \arrow[dr, phantom, "\circlearrowright"] & \prod_\tau \mathbb{R} \arrow[d, "Tr := \text{sum of the coor.}"] \\
\mathbb{Q}^\times \arrow[r] & \mathbb{C}^\times \arrow[r, "l"] & \mathbb{R}
\end{tikzcd}
$$

[Check they know this term.]

The gen F of $Gal(\mathbb{C}/\mathbb{R})$ acts on

$\mathbb{C}^\times$: usual way

$K_{\mathbb{C}}^*, \prod_\tau \mathbb{R}$: as subsets of $K_{\mathbb{C}}$.

$\mathbb{Q}^\times, \mathbb{R}, K^\times$: trivially

The maps in the diagram respect the F-action, e.g. $l \circ F = F \circ l$. [morphisms of $Gal(\mathbb{C}/\mathbb{R})$-modules]

Thus if we look at the F-fixed pts, we still have maps

defined as the F-fixed pts of $K_{\mathbb{C}}^{\vee}$ and $\prod_{\tau} \mathbb{R}$

$$\begin{tikzcd}
K^{\times} \arrow[r, "j"] \arrow[d, "N_{K/\mathbb{Q}}"'] & K_{\mathbb{R}}^{\times} \arrow[r, "l"] \arrow[d, "N"] & \left[\prod_{\tau} \mathbb{R}\right]_F \arrow[d, "tr"] \\
\mathbb{Q}^{\times} \arrow[r] & \mathbb{R}^{\times} \arrow[r, "l"] & \mathbb{R}
\end{tikzcd}$$

<u>Eventual Point</u>: $l(j(\mathcal{O}^{\times}))$ is a complete lattice in $\ker(tr)$.

Let $\rho_1, \ldots, \rho_r$ be the real emb of K,

$\sigma_1, \bar{\sigma}_1, \ldots, \sigma_s, \bar{\sigma}_s$ the comp. embeddings

Explicitly, if $\vec{v} = (\underbrace{x_1, \ldots, x_r}_{\text{cor to } \rho_i}, \overset{\text{cor to } \sigma_1}{y_1}, \overset{\text{to } \bar{\sigma}_1}{z_1}, \ldots, y_s, z_s) \in \prod_{\tau} \mathbb{R}$

then $F(\vec{v}) = (x_1, \ldots, x_r, z_1, y_1, \ldots, z_s, y_s)$.

So

$$\left[\prod_{\tau} \mathbb{R}\right]_F \cong \mathbb{R}^{r+s}$$

$$\vec{v} \longmapsto (x_1, \ldots, x_r, y_1, \ldots, y_s)$$

Set

$$\mathcal{O}_K^\times = \{\alpha \in \mathcal{O}_K \mid \mathcal{N}_{K/\mathbb{Q}}(\alpha) = \pm 1\}$$

$$S = \{\vec{x} \in K_{\mathbb{R}}^\times \mid \mathcal{N}(\vec{x}) = \pm 1\}$$

$$H = \ker(tr : [\prod_\tau \mathbb{R}]_F \to \mathbb{R})$$

We have

$$\mathcal{O}_K^\times \xrightarrow{j} S \xrightarrow{l} H$$

and set $\Gamma = l(j(\mathcal{O}_K^\times))$ an additive subgp of $H \cong \mathbb{R}^{r+s-1}$.

<u>Prop</u>: $\Gamma \cong \mathcal{O}_K^\times / \mu(K)$

<u>Pf</u>: Set $\lambda = l \circ j : \mathcal{O}_K \to \Gamma$. Need $\ker \lambda = \mu(K)$.

If $\zeta \in \mu(K)$, then $|\tau(\zeta)| = 1$ for all τ and so $\lambda(\zeta) = 0$.

Conversely, suppose $\varepsilon \in \ker \lambda$. Then $|\tau \varepsilon| = 1$ for all τ.

Thus $j(\varepsilon)$ lies in a bounded part of $j(\mathcal{O}_K) \subseteq K_{\mathbb{R}}^\times \subseteq K_{\mathbb{R}}$

As $j(\mathcal{O}_K)$ is a lattice in $K_{\mathbb{R}}$, this implies that $\ker \lambda$ is finite $\Rightarrow \ker \lambda \subseteq \mu(K)$. ▨