

Lecture 23: Units of integer rings

K a number field. Consider the units of \mathcal{O}_K :

$$\mathcal{O}_K^\times = \{ \alpha \in \mathcal{O}_K \mid \exists \beta \in \mathcal{O}_K \text{ with } \alpha\beta = 1 \} = \{ \alpha \in \mathcal{O}_K \mid \alpha^{-1} \in \mathcal{O}_K \}$$

$$= \{ \alpha \in \mathcal{O}_K \mid N_{K/\mathbb{Q}}(\alpha) = \pm 1 \}$$

$\alpha \cdot \beta$ for some $\beta \in \mathcal{O}_K$

Exs: ① $K = \mathbb{Q}(i)$, $\mathbb{Z}[i]^\times = \{1, -1, i, -i\}$ $\left/ \begin{matrix} \frac{1}{2} - \frac{\sqrt{3}}{2}i & \frac{1}{2} + \frac{\sqrt{3}}{2}i \\ & \parallel \\ & \frac{1}{2} + \frac{\sqrt{3}}{2}i \end{matrix} \right.$

② $K = \mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\zeta_3)$, then $\mathcal{O}_K^\times = \{1, -1, \zeta_3, \bar{\zeta}_3, \zeta_6, \bar{\zeta}_6\}$

[In both these cases, the units have a geom. interpretation: they're the lattice pts of \mathcal{O}_K on the unit circle.]

③ $K = \mathbb{Q}(\sqrt{2})$, $\mathcal{O}_K = \mathbb{Z}[\sqrt{2}]$. Then $\mathcal{O}_K^\times \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$ which is plausible since $(1+\sqrt{2})(-1+\sqrt{2}) = 1$. $\langle -1 \rangle \langle 1+\sqrt{2} \rangle$

Connection to classical Diophantine equations: $x + \sqrt{2}y$ for $x, y \in \mathbb{Z}$ is a unit iff $x^2 - 2y^2 = \pm 1$.

Consider

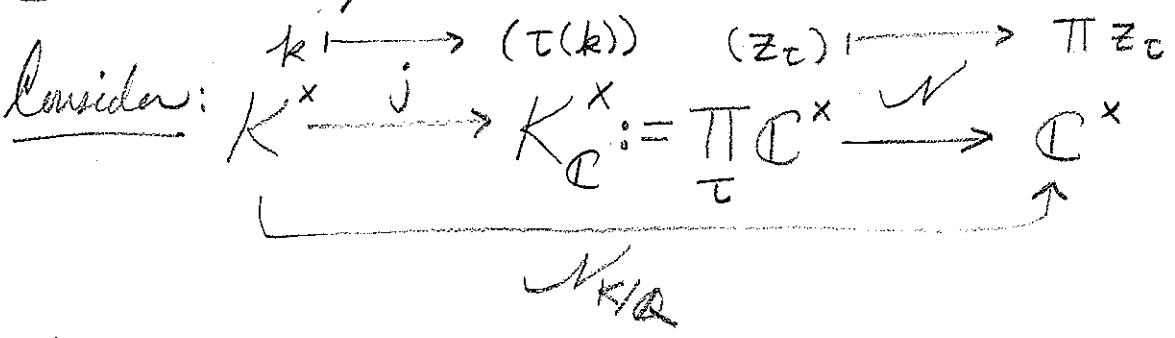
$$u(K) = \{ \alpha \in \mathcal{O}_K^\times \mid \alpha \text{ has finite order, i.e. } \alpha^m = 1 \text{ for some } m \} = \{ \text{roots of unity in } K \}$$

which is finite as there finitely many $\mathbb{Q}(\zeta_m)$ of bounded degree. More precisely, $u(K) \cong \mathbb{Z}/M\mathbb{Z}$ where M is the largest m s.t. $\mathbb{Q}(\zeta_m) \subseteq K$.

Dirichlet's Unit Theorem: $O_K^\times \cong u(K) \oplus \mathbb{Z}^{r+s-1}$.

[where $r = \#$ of real emb.,
 $s = \#$ of pairs of comp emb.]

Tool: multiplicative Minkowski:



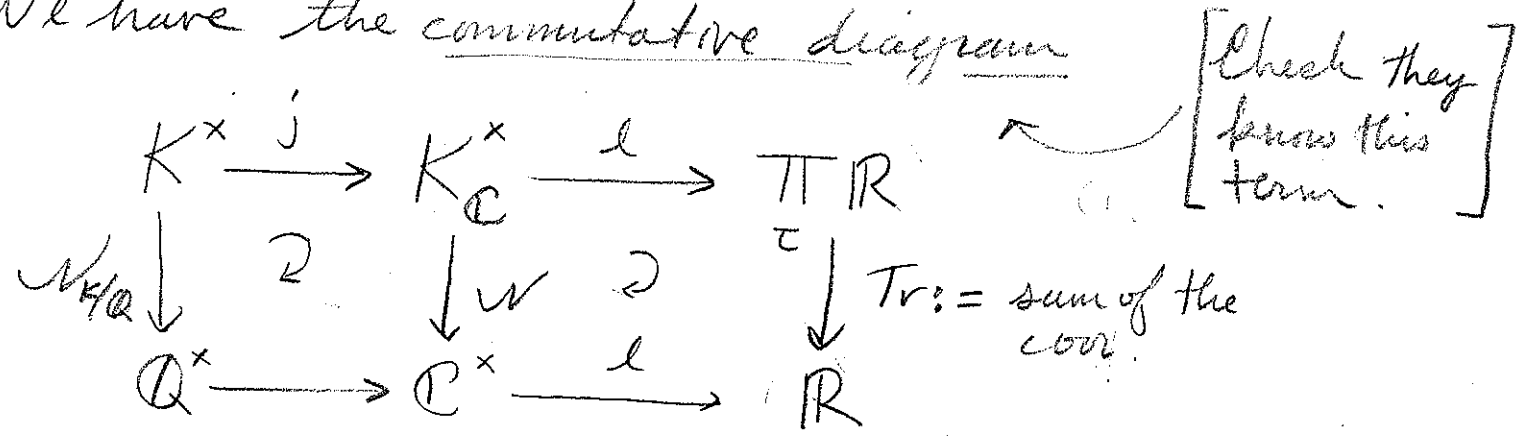
Note:
 $\mathbb{C}^\times = \mathbb{C} - \{0\}$

To get a lattice out of O_K^\times , we'll take logs as follows.

Define $l: \mathbb{C}^\times \rightarrow \mathbb{R}^\times$ by $l(z) = \log|z|$

which induces $l: K_{\mathbb{C}}^\times \rightarrow \prod_{\tau} \mathbb{R}$ coordinate-wise.

We have the commutative diagram



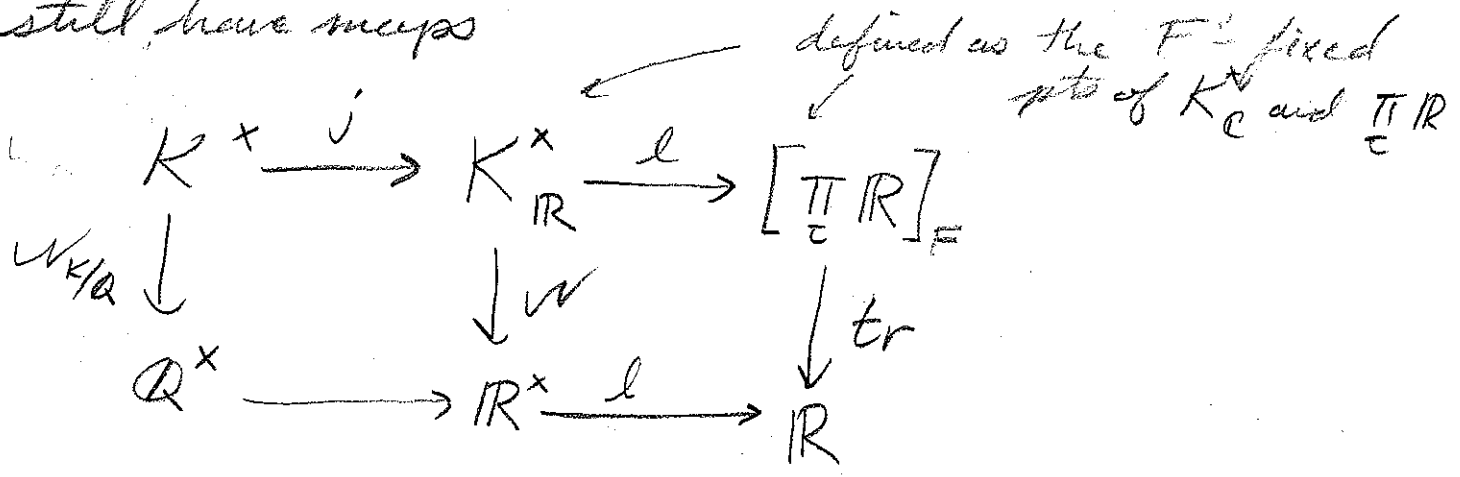
The group F of $\text{Gal}(\mathbb{C}/\mathbb{R})$ acts on

\mathbb{C}^\times : usual way $K_{\mathbb{C}}^\times, \prod_{\tau} \mathbb{R}$: as subsets of $K_{\mathbb{C}}$.

$\mathbb{Q}^\times, \mathbb{R}, K^\times$: trivially.

The maps in the diagram respect the F -action, e.g. $l \circ F = F \circ l$. [morphisms of $\text{Gal}(\mathbb{C}/\mathbb{R})$ -modules]

Thus if we look at the F -fixed parts, we still have maps



Eventual Point: $l(j(\mathbb{O}^{\times}))$ is a complete lattice in $\ker(\text{tr})$.

Let p_1, \dots, p_r be the real emb of K ,
 $\sigma_1, \bar{\sigma}_1, \dots, \sigma_s, \bar{\sigma}_s$ the comp. embeddings

Explicitly, if $\vec{v} = (\underbrace{x_1, \dots, x_r}_{\text{cor to } p_i}, \overset{\text{cor to } \sigma_1}{y_1}, \overset{\text{to } \bar{\sigma}_1}{z_1}, \dots, y_s, z_s) \in \prod_{\mathbb{C}} \mathbb{R}$

then $F(\vec{v}) = (x_1, \dots, x_r, z_1, y_1, \dots, z_s, y_s)$.

So

$$\left[\prod_{\mathbb{C}} \mathbb{R} \right]_F \cong \mathbb{R}^{r+s}$$

$$\vec{v} \longmapsto (x_1, \dots, x_r, y_1, \dots, y_s)$$

Set

$$\mathcal{O}_K^\times = \{ \alpha \in \mathcal{O}_K \mid \mathcal{N}_{K/\mathbb{Q}}(\alpha) = \pm 1 \}$$

$$S = \{ \vec{x} \in K_{\mathbb{R}}^\times \mid \mathcal{N}(\vec{x}) = \pm 1 \}$$

$$H = \ker(\text{tr}: [\mathbb{T}_{\mathbb{C}}\mathbb{R}]_F \rightarrow \mathbb{R})$$

We have

$$\mathcal{O}_K^\times \xrightarrow{j} S \xrightarrow{\ell} H$$

and set $\Gamma = \ell(j(\mathcal{O}_K^\times))$ an additive subgroup of

$$H \cong \mathbb{R}^{r+s-1}$$

Prop: $\Gamma \cong \mathcal{O}_K^\times / u(K)$

Pf: Set $\lambda = \ell \circ j: \mathcal{O}_K \rightarrow \Gamma$. Need $\ker \lambda = u(K)$.

If $\xi \in u(K)$, then $|\tau(\xi)| = 1$ for all τ and so $\lambda(\xi) = 0$.

Conversely, suppose $\varepsilon \in \ker \lambda$. Then $|\tau \varepsilon| = 1$ for all τ .

Thus $j(\varepsilon)$ lies in a bounded part of $j(\mathcal{O}_K) \subseteq K_{\mathbb{R}}^\times \subseteq K_{\mathbb{R}}$

As $j(\mathcal{O}_K)$ is a lattice in $K_{\mathbb{R}}$, this

implies that $\ker \lambda$ is finite $\Rightarrow \ker \lambda \subseteq u(K)$. \square