Lecture 20: Geometry of Numbers

HW #6: Marcus Ch 4: 17, 27
Fri Mar 20

Recall: K a number field, $n = [K:\mathbb{Q}]$

$$j: K \longrightarrow K_{\mathbb{C}} := \prod_{\tau} \mathbb{C}$$

$$k \longmapsto (\tau(k)) \qquad \tau \leftarrow \text{embeddings } K \to \mathbb{C}$$

F gen of $Gal(\mathbb{C}/\mathbb{R})$ acts on $K_{\mathbb{C}}$. Explicitly, if $\rho_1, \ldots, \rho_r : K \to \mathbb{R}$ are the real embeddings, and $\sigma_1, \bar{\sigma}_1, \ldots, \sigma_s, \bar{\sigma}_s : K \to \mathbb{C}$ are the complex embeddings, then

cor to the ρ_i — cor to σ_1 — cor to $\bar{\sigma}_1$

$$F(x_1, \ldots, x_r, y_1, z_1, \ldots, y_s, z_s)$$
$$= (\bar{x}_1, \ldots, \bar{x}_r, \bar{z}_1, \bar{y}_1, \ldots, \bar{z}_s, \bar{y}_s)$$

Def: $K_{\mathbb{R}}$ = pts of $K_{\mathbb{C}}$ fixed by F.

Prop: $K_{\mathbb{R}} \longrightarrow \mathbb{R}^r \oplus \mathbb{C}^s$ given by

$$(x_1, \ldots, x_r, y_1, z_1, \ldots) \longmapsto (x_1, \ldots, x_r, y_1, y_2, \ldots, y_s)$$

is an isomorphism. [Can think of $K_{\mathbb{R}}$ as defined by]

$K_{\mathbb{C}}$ has a hermian inner product:

$$\langle \vec{x}, \vec{y} \rangle = \sum_\tau x_\tau \overline{y_\tau} \qquad \mathbb{C}\text{-linear in } \vec{x},$$

$$\langle \vec{x}, \vec{y} \rangle = \overline{\langle \vec{y}, \vec{x} \rangle} \quad [\Rightarrow \mathbb{C}\text{-anti-linear in } \vec{y}.]$$

Notice $\langle F(\vec{x}), F(\vec{y}) \rangle = F(\langle \vec{x}, \vec{y} \rangle)$, so $\langle\ ,\ \rangle$ restricts to a symmetric bilinear form $K_{\mathbb{R}} \times K_{\mathbb{R}} \to \mathbb{R}$, called the <u>canonical metric</u>, namely if

$$K_{\mathbb{R}} \cong \mathbb{R}^r \oplus \mathbb{C}^s$$

then $\langle\ ,\ \rangle$ is the usual inner product on $\mathbb{R}^r$ and $2\times$ it on $\mathbb{C}^s$.

<u>Ex</u>: $K = \mathbb{Q}(\alpha)$, $\alpha^2 = -1$ $\qquad \sigma_1 : K \to \mathbb{C}$, $\alpha \mapsto i$ $\qquad \overline{\sigma_1} : K \to \mathbb{C}$, $\alpha \mapsto -i$

$$K_{\mathbb{C}} = \mathbb{C} \times \mathbb{C} \qquad F(z, w) = (\overline{w}, \overline{z})$$

$$K_{\mathbb{R}} = \{ (z, \overline{z}) \in K_{\mathbb{C}} \} \longrightarrow \mathbb{C}$$

$$(z, \overline{z}) \longmapsto z$$

$$\begin{aligned} \langle z, w \rangle &= \langle (z,\overline{z}), (w, \overline{w}) \rangle \\ &= z\overline{w} + \overline{z}w \\ &= (ac+bd) + (ac+bd) \\ &= 2((a,b)\cdot(c,d)) \end{aligned}$$

if $z = a+bi$
$w = c+di$

think $\mathbb{Z}^k$ in $\mathbb{R}^k$.

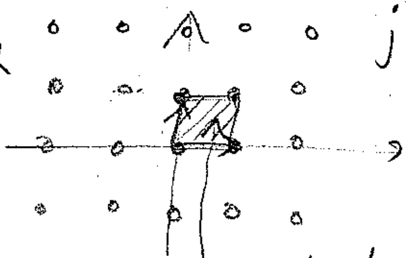
Thm: $j(\mathcal{O}_K)$ is a complete lattice in $K_{\mathbb{R}}$.

Moreover $\mathrm{Vol}(j(\mathcal{O}_K)) = \mathrm{Vol}\left(K_{\mathbb{R}}/j(\mathcal{O}_K)\right) = \sqrt{|\Delta_K|}$

w.r.t. the cannonical metric

Ex: $K = \mathbb{Q}(i)$

$K_{\mathbb{R}}$:

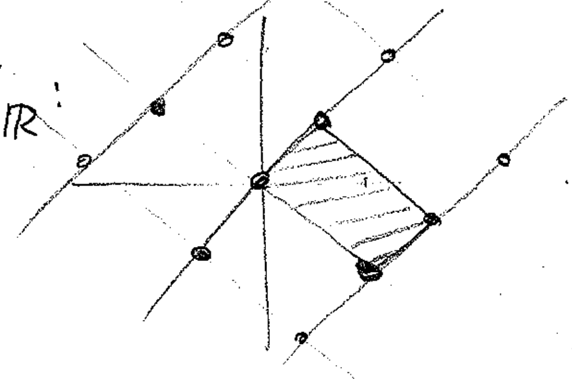


$j(\mathcal{O}_K = \mathbb{Z}[i])$

side lengths $= \sqrt{2}$ w.r.t to $\langle , \rangle$ $\Rightarrow$ Area $= 2$, $\Delta_K = -4$

Ex: $K = \mathbb{Q}(\sqrt{2})$

$K_{\mathbb{R}}$:



Area $= \sqrt{2} \cdot 2$

$\Delta_K = 8$

Def: A subgroup $\Gamma \leq \mathbb{R}^m$ is a complete lattice if $\exists$ a basis $v_1, \dots, v_m$ of $\mathbb{R}^m$ so that

$$\Gamma = \{\textstyle\sum a_i v_i \mid a_i \in \mathbb{Z}\}$$

Prop: $\Gamma \leq \mathbb{R}^m$ is a complete lattice iff Γ is discrete (every pt of Γ is isolated) and $\mathbb{R}^m/\Gamma$ is compact.

Pf: ($\Rightarrow$) In the right basis $\Gamma = \mathbb{Z}^m \leq \mathbb{R}^m$ which is clearly discrete and $\mathbb{R}^m/\mathbb{Z}^m = (S^1)^m$ is cpt.

($\Leftarrow$) As $\mathbb{R}^m/\Gamma$ is cpt, Γ must contain a basis $\{v_1, \dots, v_m\}$. Let $\Gamma_0 \leq \Gamma$ be the subgp gen by $\{v_i\}$. Now Γ/Γ_0 is discrete in $\mathbb{R}^m/\Gamma_0$ which is cpt $\Rightarrow \Gamma/\Gamma_0$ is finite, say $m = [\Gamma : \Gamma_0]$. Then

$$\Gamma_0 \leq \Gamma \leq \frac{1}{m}\Gamma_0 \Rightarrow \Gamma \cong \Gamma_0 \cong \mathbb{Z}^m.$$

$\blacksquare$

Ex: $SL_2\mathbb{Z}$ in $SL_2\mathbb{R}$ is also a lattice = discrete + finite covolume.

Pf of thm: Let $\alpha_1, \dots, \alpha_n$ be an int basis for $\mathcal{O}_K$.

Set $\vec{v}_i = j(\alpha_i) = (\tau(\alpha_i)) \in K_{\mathbb{R}}$

Consider the Gram matrix

$$G = (\vec{v}_i \cdot \vec{v}_j) = \left(\sum_\tau \tau(\alpha_i \alpha_j)\right) = \left(tr_{K/\mathbb{Q}}(\alpha_i \alpha_j)\right)$$

Thus

$$\Delta_K \underset{\substack{\uparrow \\ \text{by def}}}{=} \det G \underset{\substack{\uparrow \\ \text{usual reason.}}}{=} \mathrm{Vol}(j(\mathcal{O}_K))^2$$

$\blacksquare$