Lecture 7:

Last time:

Thm: $\mathcal{O}$ a Dedekind domain. Then any ideal $\mathfrak{a} = \mathfrak{p}_1\mathfrak{p}_2\cdots\mathfrak{p}_n$ with $\mathfrak{p}_i$ prime, unique up to order.

Lemma: $\mathfrak{p}$ a prime ideal of $\mathcal{O}$. Set $\mathfrak{p}^{-1} = \{k \in K \mid k\mathfrak{p} \subseteq \mathcal{O}\}$. Then $\mathfrak{a}\mathfrak{p}^{-1} \neq \mathfrak{a}$ for any ideal $\mathfrak{a} \neq 0$.

Cor: $\mathfrak{p}$ prime. Then $\mathfrak{p}^{-1}\mathfrak{p} = \mathcal{O}$.

Pf: We have

$$\mathfrak{p} \underset{\text{by lemma}}{\subsetneq} \mathfrak{p}^{-1}\mathfrak{p} \overset{\text{by def of } \mathfrak{p}^{-1}}{\subseteq} \mathcal{O}$$

As $\mathfrak{p}$ is maximal, must have $\mathfrak{p}^{-1}\mathfrak{p} = \mathcal{O}$. ▨

Pf of Thm: (Existence) Let $\mathfrak{a}$ be a maximal elt of

$$\mathcal{M} = \left\{ \begin{array}{l} \text{ideals not } (0) \text{ or } (1) \\ \text{without prime decomp} \end{array} \right\}$$

Pick a maximal $\mathfrak{p} \supsetneq \mathfrak{a}$. Now

$$\mathfrak{a} \underset{\substack{\uparrow \\ \text{By lemma}}}{\subsetneq} \mathfrak{a}\mathfrak{b}^{-1} \subsetneq \mathfrak{b}\mathfrak{b}^{-1} = \mathcal{O}$$

otherwise $\mathfrak{b} = (\mathfrak{a}\mathfrak{b}^{-1})\mathfrak{b} = \mathfrak{a}\mathcal{O} = \mathfrak{a}$

So $\mathfrak{a}\mathfrak{b}^{-1}$ is a proper ideal not in $\mathcal{M}$ $\Rightarrow$ has a prime factorization. But then $\mathfrak{a} = (\mathfrak{a}\mathfrak{b}^{-1})\mathfrak{b}$ has one as well, a contradiction. <u>Uniqueness</u>: on HW. ▨

<u>Def</u>: A <u>fractional ideal</u> of K is a finitely generated $\mathcal{O}$-submodule.

<u>Ex</u>: $a \in K$, consider $(a) = a\mathcal{O}$. <u>Ex</u>: an ideal of $\mathcal{O}$ is just a fractional ideal $\subseteq \mathcal{O}$. called "integral ideals"

<u>Ex</u>: $K = \mathbb{Q}$. $\mathfrak{a} = \left\{ \frac{5n}{3} \mid n \in \mathbb{Z} \right\}$

If $\mathfrak{a} \subseteq K$ is a fractional ideal, $\exists\, c \in \mathcal{O}$ such that $c\mathfrak{a}$ is an ideal of $\mathcal{O}$. Thus $\mathfrak{a} = \frac{1}{c}\mathfrak{b}$, where $\mathfrak{b}$ is an ideal of $\mathcal{O}$.

<u>Def</u>: The <u>ideal group</u> of K is $J_K = \left\{ \begin{matrix} \text{all fractional} \\ \text{ideals of } K \end{matrix} \right\}$

<u>operation</u>: mult of ideals <u>identity</u>: $(1) = \mathcal{O}$.

<u>inverses</u>: $\mathfrak{a}^{-1} = \{ k \in K \mid k\mathfrak{a} \subseteq \mathcal{O} \}$

Proof that inverses work. Clearly $\mathfrak{a}\mathfrak{a}^{-1} \subseteq \mathcal{O}$

Write $\mathfrak{a} = \frac{1}{c}\mathfrak{p}_1 \cdots \mathfrak{p}_n$ where $c \in \mathcal{O}$ and $\mathfrak{p}_i \subseteq \mathcal{O}$ are prime

Let $\mathfrak{b} = c\,\mathfrak{p}_1^{-1} \cdots \mathfrak{p}_n^{-1}$. Then $\mathfrak{a}\mathfrak{b} = \mathcal{O}$ and so $\mathfrak{b} \subseteq \mathfrak{a}^{-1}$ $\blacksquare$

and $\mathfrak{a}^{-1}\mathfrak{a} = \mathcal{O}$.

If $\mathfrak{a}$ is a fractional ideal, $\mathfrak{a} = \mathfrak{c}^{-1}\mathfrak{b}$ with $\mathfrak{c}, \mathfrak{b}$ integral

So $\mathfrak{a} = \mathfrak{p}_1^{v_1} \cdots \mathfrak{p}_n^{v_n}$ where $\mathfrak{p}_i \subseteq \mathcal{O}$ are prime and $v_i \in \mathbb{Z}$.

So $J_K = \bigoplus_{\substack{\text{prime} \\ \text{ideals of } \mathcal{O}}} \mathbb{Z}$. Eg. $\mathbb{Q}^\times = \bigoplus_{\substack{\text{primes} \\ p}} \mathbb{Z}$

$P_K \leq J_K$ the subgroup of principal ideals

<u>Def</u>: The <u>class group</u> of K is $Cl_K = J_K / P_K$

<u>Note</u>: Every elt of Cl_K can be rep. by an <u>integral</u> ideal. Equiv relation on such is

$[\mathfrak{a}] = [\mathfrak{b}]$ if $\exists\, \alpha, \beta \in \mathcal{O}$ with $\alpha\mathfrak{a} = \beta\mathfrak{b}$.

This is how the book defines it.

Point: $Cl_K = 1 \iff \mathcal{O}$ is a P.I.D. $\iff \mathcal{O}$ is a U.F.D.

as $\mathcal{O}$ is a Dedekind domain.

Moral: Measures failure of unique factorization

Will show

$h = |Cl_K| < \infty$ is called the class number of K.