Lecture 13: Hilbert's Ramification Theory

L/K normal, $\mathfrak{p}$ a prime of $\mathcal{O}_K$.

$\mathrm{Gal}(L/K)$ acts on $\left\{\begin{smallmatrix}\text{primes of } \mathcal{O}_L \\ \text{above } \mathfrak{p}\end{smallmatrix}\right\} = \mathcal{Q}$ e.g.

have a hom. $\mathrm{Gal}(L/K) \to \mathrm{Sym}(\mathcal{Q})$

Last time: Action is transitive.

Will use this action to L/K up into simpler pieces from $\mathfrak{p}$'s point of view: $\mathfrak{p}\mathcal{O}_L = \mathfrak{q}_1^e \cdots \mathfrak{q}_r^e$ $f_i = f$

ram index   inertial degree

$$\begin{array}{llllcc}
 & L & \mathfrak{q} & & & \\
 & | & & \Big\rangle & e & 1 \\
\text{interial field} & L_E & \mathfrak{q}_E & & & \\
 & | & & \Big\rangle & 1 & f \\
\text{decomposition field} & L_D & \mathfrak{q}_D & & & \\
 & | & & \Big\rangle & 1 & 1 \\
 & K & \mathfrak{p} & & &
\end{array}$$

Here $\mathfrak{q}_E = \mathfrak{q} \cap \mathcal{O}_{L_E}$ and $\mathfrak{q}_D = \mathfrak{q} \cap \mathcal{O}_{L_D}$

Set $G = \mathrm{Gal}(L/K)$.

Stabilizer of $\mathfrak{q}$ under G-action

$D =$ Decomposition group $= \{\sigma \in G \mid \sigma(\mathfrak{q}) = \mathfrak{q}\} = G_{\mathfrak{q}}$

$E = $ Inertia Group $= \{\sigma \in D \mid \sigma$ acts trivially on $\mathcal{O}_L/\mathfrak{q}\}$

Then $L_E =$ fixed field of E $= \{\alpha \in L \mid \sigma(\alpha) = \alpha$ for all $\sigma \in E\}$

$$\begin{array}{c} L_E \\ | \\ L_D \end{array}$$

$L_D =$ fixed field of D.

<u>Notation</u>: $(\text{blah})_E$ = things fixed by E

$$(\mathcal{O}_L)_E = \mathcal{O}_L \cap L_E = \mathcal{O}_{L_E} \quad \text{and} \quad \mathfrak{q}_E = \mathfrak{q} \cap \mathcal{O}_{L_E}$$

<u>Thm</u>: (i) $\mathfrak{q}_D$ is non-split in L_D
[i.e. $\mathfrak{q}$ is the <u>only</u> prime above $\mathfrak{q}_D$]

(ii) $\mathfrak{q}$ over $\mathfrak{q}_D$ has ram index $= e$
inertial degree $= f$

(iii) $\mathfrak{q}_D$ over $\mathfrak{p}$ has ram index $= 1$
inertial deg $= 1$

$$\begin{array}{cc} L & \mathfrak{q} \\ | & \\ L_D & \mathfrak{q}_D \\ | & \\ K & \mathfrak{p} \end{array}$$

<u>Cor</u>: In L_D, $\mathfrak{p}$ breaks up into r pieces (same as in L) unramified, with residue field $(= \mathcal{O}_{L_D}/\mathfrak{q}_D$ or $\mathcal{O}_K/\mathfrak{p})$ unchanged. Also $[L_D : K] = r$. That is, $\mathfrak{p}$ is totally split in L_D.

<u>Proof</u>: (i) $\mathrm{Gal}(L/L_D) = D$ acts <u>transitively</u> on the primes above $\mathfrak{q}_D$ yet fixes $\mathfrak{q}$.

L/K is normal, hence so is L/L_D.

(cont) As L/K is normal, we have

$n = [L:K] = \sum e_i f_i = efr$. Since $|D| r = |G|$,

we have $|D| = ef$. Thus, . By (i) $\mathfrak{p}$ must split into r factors in $\mathcal{O}_{L_D}$ so the fund. ident forces $e'' = f'' = 1$. By mult. of ram index + inertial deg, get $e' = e$ and $f' = f$. ▨

$$\begin{array}{ll} L & \\ \mid & \deg = |D| = ef \ (e', f') \\ L_D & \\ \mid & \deg = r \ (e'', f'') \\ K & \end{array}$$

Consider the residue fields

$$\begin{array}{c} K(\mathfrak{q}) = \mathcal{O}_L/\mathfrak{q} \\ \mid \\ K(\mathfrak{p}) = \mathcal{O}_K/\mathfrak{p} \end{array}$$

<u>Prop:</u> This extension is normal, and $D \longrightarrow \mathrm{Gal}\left(K(\mathfrak{q})/K(\mathfrak{p})\right)$ is surjective.

<u>Proof:</u> By the above, $K(\mathfrak{p}) \cong K(\mathfrak{q}_D)$ so we might as well assume $L_D = K \Rightarrow D = G$.

Blah, blah, ...