Lecture 37:

<u>Last time</u>: V = Places of $\mathbb{Q}$ = $\{\text{primes}\} \cup \{\infty\}$

<u>Thm</u>: Let $a_1, \dots, a_k \in \mathbb{Q}^\times$ and choose $\varepsilon_{i,v} = \pm 1$ for $1 \le i \le k$ and $v \in V$. Then $\exists\, b \in \mathbb{Q}^\times$ with $(a_i, b) = \varepsilon_{i,v}$ for all i, v if and only if

① Almost all $\varepsilon_{i,v} = 1$.

② $\prod_v \varepsilon_{i,v} = 1$ for each i.

③ $\exists\, x_v \in \mathbb{Q}_v^\times$ s.t. $(a_i, x_v) = \varepsilon_{i,v}$ for all i.

Last time, I wrote "$\exists\, x_{i,v} \in \mathbb{Q}_v^\times \dots$"

Set $S = \{2, \infty\} \cup \{\text{primes dividing some } a_i\}$

$T = \{v \in V \mid \varepsilon_{i,v} = -1 \text{ for some } i\}$

We showed the thm holds when $S \cap T = \emptyset$.

[General case will use:]

<u>Approximation Thm</u>: S a finite subset of V.

The map $\mathbb{Q} \longrightarrow \prod_{v \in S} \mathbb{Q}_v$ has dense image,

$$q \longmapsto (q, q, \dots, q)$$

where the range has the product topology.

(i.e. basic open set $= \prod_{v \in S} U_v$ where $U_v \subseteq \mathbb{Q}_v$ is open.)

Ex: $S = \{\infty\}$. Then Approx says $\mathbb{Q}$ is dense in $\mathbb{R}$.

Rmk: $\mathbb{Q} \longrightarrow \prod_{v \in V} \mathbb{Q}$ has discrete image.

[Compare to Minkowski geometry, foreshadow adèles and idèles. Follows from the Chinese Rem. Thm.]

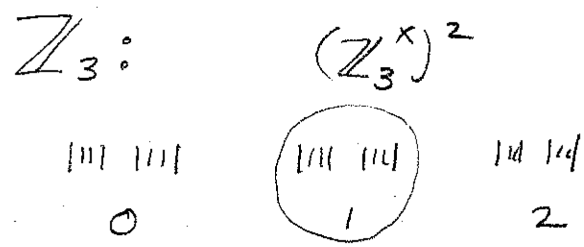
Lemma: $(\mathbb{Q}_v^\times)^2$ is an open subset of $\mathbb{Q}_v$.

Proof: $v = \infty$: $(\mathbb{R}^\times)^2 = \{x \in \mathbb{R}^\times \mid x > 0\}$ is open in $\mathbb{R}$.

$v = p$ odd: First, $(\mathbb{Z}_p^\times)^2$ is open in $\mathbb{Z}_p$ since $(\mathbb{Z}_p^\times)^2 =$ inverse image of $(\mathbb{F}_p^\times)^2$ under

$$\mathbb{Z}_p \longrightarrow \mathbb{F}_p.$$

$$= \bigcup_{\substack{1 \le a \le p-1 \\ a \text{ a square mod } p}} B_1(a)$$



As $\mathbb{Z}_p$ is open in $\mathbb{Q}_p = \bigcup_{k \in \mathbb{Z}} p^k \mathbb{Z}_p^\times$, $(\mathbb{Z}_p^\times)^2$ is open in $\mathbb{Q}_p$ and so

$$(\mathbb{Q}_p^\times)^2 = \bigcup_{k \in \mathbb{Z}} \underbrace{p^{2k} (\mathbb{Z}_p^\times)^2}_{\substack{\text{open as mult} \\ \text{by } p^{2k} \text{ is a homeo}}} \quad \text{is open.}$$

$v = 2$: Exercise. ▨

<u>Proof of Thm</u>: Recall defs of S + T.

<u>Claim</u>: $\exists\, x' \in \mathbb{Q}^\times$ s.t. $x' x_v$ is a square in $\mathbb{Q}_v^\times$ for $v \in S$.

Set $\eta_{i,v} = \varepsilon_{i,v}(a_i, x')_v$, which sat hyps ①, ② and ③ of the thm, since $(a_i, x' x_v)_v = (a_i, x_v)(a_i, x')_v = \eta_{i,v}$.

<u>Key</u>: For $v \in S$, $\eta_{i,v} = (a_i, x' x_v)_v = 1$.

So we're in the case where $S \cap T = \emptyset$. So $\exists\, b' \in \mathbb{Q}^\times$ with $(a_i, b')_v = \eta_{i,v}$ for all i, v. Then

$b = x' b'$ is the needed sol'n since

$$(a_i, b)_v = (a_i, x')_v (a_i, b')_v = \varepsilon_{i,v} (a_i, x')_v^2 = \varepsilon_{i,v}.$$

<u>Pf of Claim</u>: $\mathcal{U} = \prod_{v \in S} \underbrace{x_v \cdot (\mathbb{Q}_v^\times)^2}_{\text{open in } \mathbb{Q}_v}$ is open in $\prod_{v \in S} \mathbb{Q}_v$

So by the Approx Thm, $\exists\, x'$ s.t. $(x', \dots, x') \in \mathcal{U}$,

i.e. for $v \in S$ we have $x' = x_v y_v^2 \Rightarrow x' x_v \in (\mathbb{Q}_v^\times)^2$.

This proves the thm, modulo the Approx. Thm. ▨

Proof of Approximation: Suppose $S = \{\infty, p_1, \dots, p_k\}$ (enlarging S only makes things harder) Need to show that each open set $U = \prod_{v \in S} B_\delta(x_v)$ contains a point of (the image of) $\mathbb{Q}$. That is, need $x \in \mathbb{Q}$ s.t. $|x - x_\infty|_\infty < \delta$ and $|x - x_i|_{p_i} < \delta$ for all i. Choose N s.t. $p_i^{-N} < \delta$ for all i.

By the Chinese Remainder Thm, $\exists\, y \in \mathbb{Z}$ with $y \equiv x_i \bmod p_i^N$ for all i. Then $|y - x_i|_{p_i} \leq p_i^{-N} < \delta$.

Let q be a prime not any of the p_i. Choose $a, n \in \mathbb{Z}$ s.t.

$$\left| (x_\infty - y) - \underbrace{\frac{a\, p_1^N \cdots p_k^N}{q^n}}_{\substack{\text{dense in } \mathbb{R} \\ \text{as } a, n \text{ vary}}} \right|_\infty < \delta.$$

Then $x = y + \dfrac{a\, p_1^N \cdots p_k^N}{q^n}$ is in U, as desired. $\blacksquare$