Lecture 36: Local to Global for Hilbert's Symbol

Last time: [First, complete the proof of:]

Thm: $a, b \in \mathbb{Q}^\times$. Then all but finitely many $(a,b)_v = 1$ and $\prod_v (a,b)_v = 1$.

Fix $a \in \mathbb{Q}^\times$. Suppose given $\{\varepsilon_v = \pm\}$. Does $\exists\, b \in \mathbb{Q}^\times$ with $(a,b)_v = \varepsilon_v$ for all v? Need some cond on $\{\varepsilon_v\}$:

① $\underbrace{\text{Almost all}}$ $\varepsilon_v = 1$
└ means "all but finitely many"

② $\prod_v \varepsilon_v = 1$

Need more than this since $(4,b) = 1$ for all $b \in \mathbb{Q}^\times$, and $(2,b)_7 = 1$ for all $b \in \mathbb{Q}_2^\times$ (as $4^2 = 16 \equiv 2 \bmod 7$).

Thm: Let $a_1, \dots, a_k \in \mathbb{Q}^\times$ and choose $\varepsilon_{i,v} = \pm 1$ for $0 \le i \le n$ and v a place of $\mathbb{Q}$. Then $\exists\, b \in \mathbb{Q}^\times$ with $(a_i, b)_v = \varepsilon_{i,v}$ for all i, v iff

① Almost all $\varepsilon_{i,v} = 1$.

② $\prod_v \varepsilon_{i,v} = 1$ for each i.

③ $\exists\, x_{i,v} \in \mathbb{Q}_v^\times$ s.t. $(a_i, x_{i,v}) = \varepsilon_{i,v}$.

[Conditions ①, ②, and ③ are clearly necessary. To prove this theorem, which is the hypothesis of Hasse-Minkowski when $n=4$, we'll need...]

__Notation:__ $V = \left\{ \begin{smallmatrix} \text{places} \\ \text{of } \mathbb{Q} \end{smallmatrix} \right\} = \{\text{rat'l primes}\} \cup \{\infty\}$

__Approximation Thm:__ S a finite subset of V. The map $\mathbb{Q} \longrightarrow \prod_{v \in S} \mathbb{Q}_v$ has dense image,

$$q \longmapsto (q, q, \ldots)$$

where $\prod_{v \in S} \mathbb{Q}_v$ has the product topology.

Have to end

__Ex:__ $S = \{\infty\}$, thm says $\mathbb{Q}$ is dense in $\mathbb{R}$.

__Remark:__ $\mathbb{Q} \longrightarrow \prod_{v \in V} \mathbb{Q}_v$ has __discrete__ image.

[Compare to the Minkowski construction, foreshadow adeles and ideles. Approx Thm follows from C.R.T.]

__Dirichlet's Thm:__ $a, m \in \mathbb{N}$ rel. prime. There exist infinitely many rat'l primes $p \equiv a \bmod m$.

[Pf: L-functions, complex analysis.]

__Pf of Thm:__ Can take $a_i \in \mathbb{Z}$. Let

$S = \{2, \infty\} \cup \{\text{primes div the } a_i\}$ and

$T = \{v \in V \mid \varepsilon_{i,v} = -1 \text{ for some } i\}$

Case: $S \cap T = \emptyset$

[Recall: $\varepsilon, \eta \in \mathbb{Z}_p^\times$ then $(\varepsilon, \eta) = 1$ and $(\varepsilon, p) = \left(\frac{\varepsilon}{p}\right)$]

$\uparrow$ odd

Let

$$a = \prod_{\substack{v \in T \\ v \neq \infty}} v \quad \text{and} \quad m = 8 \prod_{\substack{v \in S \\ v \neq 2, \infty}} v$$

By Dirichlet, $\exists$ a prime $p \notin S \cup T$ with $p \equiv a \bmod m$.

Claim: $b = ap$ is the desired sol'n.

Case $v = \infty$: $(a_i, b)_\infty = 1$ as $b > 0$.

Case $v \in S \setminus \{2, \infty\}$: $b = ap \equiv a^2 \bmod v \Rightarrow b \in (\mathbb{Z}_v^\times)^2$

$\Rightarrow (a_i, b)_v = 1$

Case $v = 2$: $b \equiv a^2 \bmod 8 \overset{\text{gen of HW prob}}{\Longrightarrow} b \in (\mathbb{Z}_2^\times)^2 \Rightarrow (a_i, b)_2 = 1$.

Case $v \notin S \cup T \cup \{p\}$: $a_i \in \mathbb{Z}_v^\times$ and $b \in \mathbb{Z}_v^\times$ so

$(a_i, b)_v = 1$

Case $v \in T$: $a_i \in \mathbb{Z}_v^\times$ and $b \in v\mathbb{Z}_v^\times$, so

$$(a_i, b)_v = \left(\frac{a_i}{v}\right) = (a_i, x_v)_v = \varepsilon_{i,v}$$

since $x_v \in v\mathbb{Z}_v^\times$ as $\underline{\text{some}}$ $(a_i, x_v)_v \neq 1$.

Case $v = p$: Have $\prod (a_i, b)_v = 1$ and $\prod \varepsilon_{i,v} = 1$

Since $(a_i, b)_v = \varepsilon_{i,v}$ for all $v \neq p$, this forces

$(a_i, b)_p = \varepsilon_{i,p}$.

This completes the proof of the Thm when $S \cap T = \emptyset$.

General case: Next time.