Lecture 33: quadratic forms over $\mathbb{Q}_p$.

Hasse-Minkowski: A quad form q over $\mathbb{Q}$ reps 0 iff q_v reps 0 for every place v.

[$\uparrow$ form over $\mathbb{Q}_v$] [Remind of importance of isotropicness...]

Cor: For $a \in \mathbb{Q}$, q reps $a \iff q_v$ reps a for all v.

Pf: $a x_{n+1}^2 - q(x_1, \ldots, x_n)$ reps $0 \iff q$ reps a. ▨

Cor: q and q' quad. forms over $\mathbb{Q}$. They are isometric iff $q_v \cong q'_v$ for all v.

Pf: Reduce to the case where q and q' are nondegenerate. Let V and V' be the vector spaces on which q and q' are defined. We'll induct on $\dim V = \dim V'$.

Pick $x \in V$ with $q(x) = a \neq 0$. Then q_v reps a for all $v \Rightarrow q'_v$ reps a for all $v \underset{\text{Cor}}{\Rightarrow} \exists\, x'$ with $q_v'(x') = a$.

So
$$V = \langle x \rangle \hat{\oplus} \langle x \rangle^{\perp} \quad \text{and} \quad V' = \langle x' \rangle \hat{\oplus} \langle x' \rangle^{\perp}$$

Let $f = q|_{\langle x \rangle^{\perp}}$ and $f' = q'|_{\langle x' \rangle^{\perp}}$.

<u>Claim</u>: $f_v \cong f'_v$ for all places v.

If true, have $f \cong f'$ by induction $\Rightarrow q \cong q'$ as needed.

<u>Pf of Claim</u>: Have an isom $\tau_1 : (V_v, q_v) \longrightarrow (V'_v, q'_v)$

Have $q'(\tau_1(x)) = a$. <u>By <u>HW</u></u>, $\exists$ an isom τ_2 of (V'_v, q'_v) taking $\tau_1(x)$ to x'. Set $\tau = \tau_2 \circ \tau_1$, an isom $V_v \to V'_v$ taking x to x'. Then $\tau(\langle x \rangle^\perp) = \langle x' \rangle^\perp$ and so $f_v \cong f'_v$. ▨

<u>Hilbert symbol</u>: $a, b \in K^\times$

$$(a,b) = \begin{cases} 1 & \text{if } z^2 - ax^2 - by^2 = 0 \text{ has a nontrivial solution.} \\ -1 & \text{otherwise} \end{cases}$$

<u>Quad. Forms over $\mathbb{Q}_p$</u>: (V, q) nondegenerate.

Choose a basis where $q(x) = \sum_{i=1}^{n} a_i x_i^2$.

$$\operatorname{disc}(q) = \prod a_i \quad (\text{in } \mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^2)$$

$$\varepsilon(q) = \prod_{i<j} (a_i, a_j)$$

See Serre for a proof.

<u>Prop</u>: ε doesn't depend on the choice of basis.

Thm: Two nondeg quad forms on $\mathbb{Q}_p^n$ are isometric iff they have the same disc + ε.

Cor: There are at most 16 such forms as $\left|\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2\right| = 4$ or 8.

[Proof of thm similar to the one at the beginning, inducting on the dimension. The key is thus showing that q reps certain values dep only on (disc, ε). And that's determined by when certain other forms rep 0. Thus the fulcrum of the proof is]

q quad form on $\mathbb{Q}_p^n$, $d = \text{disc}(q)$ $\varepsilon = \varepsilon(q)$.
nondegenerate.

Lemma: q reps 0 iff

$n = 2$ and $d = -1$ (in $\mathbb{Q}_p^\times$/squares)
$n = 3$ and $(-1,-d) = \varepsilon$.
$n = 4$ and either $d \neq 1$ or ($d = 1$ and $\varepsilon = (-1,-1)$)
$n \geq 5$

Cor of this + HM: q a quad form on $\mathbb{Q}^n$.

If $n \geq 5$, then q reps 0 iff q_∞ reps 0.

Proof of lemma: [$n=1$ is trivial.]

$n=2$: $q = a_1 x_1^2 + a_2 x_2^2$ reps $0 \iff -\frac{a_1}{a_2}$ is a square $\iff -a_1 a_2 = -d = 1$ in $\mathbb{Q}_p^\times/\text{sqs}$.

$n=3$: $q = a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2$ reps $0 \iff$

$a_3 q \sim a_3 a_1 x_1^2 + a_3 a_2 x_2^2 + x_3^2$ reps $0 \iff$

$$1 = (-a_1 a_3, -a_2 a_3) =$$

$$(-1,-1)(a_1,-1)(-1,a_2)(a_1,a_2)(a_1,a_3)(a_3,a_2)\underbrace{(a_3,a_3)}_{(a_3,-a_3^2) = (-1,a_3)}$$

$$= (-1,-1)(-1,\underbrace{a_1 a_2 a_3}_{d})\underbrace{(a_1,a_2)(a_1,a_3)(a_2,a_3)}_{\varepsilon}$$

$$= (-1,-d)\,\varepsilon.$$

$n \geq 4$: Next time. ∎