Lecture 31: Quadratic Forms II

Last time: K field w/ char $\neq 2$.

Quadratic Space:

V - finite dim'l vector sp / K

B - sym bilinear form.

Quadratic Form: $q : V \to K$ given by $q(v) = B(v,v)$

Thm: $\exists$ a basis for V where $G = (B(e_i, e_j))$ is diagonal, i.e. $q(x) = \sum a_i x_i^2$.

Classification of non-degenerate quad forms, up to isometry ($f : V_1 \to V_2$ with $B_1(v,w) = B_2(f(v), f(w))$)

$\underline{K = \mathbb{C}}$: $(\mathbb{C}^n, x_1^2 + \cdots + x_n^2)$

$\underline{K = \mathbb{R}}$: $(\mathbb{R}^n, x_1^2 + \cdots + x_k^2 - (x_{k+1}^2 + \cdots + x_n^2))$ $\quad 0 \leq k \leq n$

signature $=$ # pos $-$ # neg $= 2k - n$

Pf: Choose a basis $\{e_1, \dots, e_n\}$ where $G = \mathrm{diag}(d_1, \dots, d_n)$. Replacing e_i with λe_i changes d_i to $\lambda^2 d_i$. ▨

Things are more complicated with a small field, like $K = \mathbb{Q}$.

Ex: $V = \mathbb{Q}^2$, $q = 8x_1^2 + 2x_1x_2 - 3x_2^2$ $\quad G = \begin{pmatrix} 8 & 1 \\ 1 & -3 \end{pmatrix}$

Saw $\begin{pmatrix} 8 & 1 \\ 1 & -3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -3 & 0 \\ 0 & 75 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -3 & 0 \\ 0 & 3 \end{pmatrix}$ [Query: Is this the simplest form.]

In fact, this form is just $x_1^2 - x_2^2$; if

$C = \frac{1}{2}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ then $C^t \begin{pmatrix} -3 & 0 \\ 0 & 3 \end{pmatrix} C = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

Point: $q(2,-1) = -3 \cdot 2^2 + 3 \cdot (-1)^2 = -9 = -1$ (a square).

On the other hand, $x_1^2 - 3x_2^2$ is really diff than $x_1^2 - 3x_2^2$ because $-3 \neq -1$ in $\mathbb{Q}^\times / (\mathbb{Q}^\times)^2$.

Isotropic Vector: $v \neq 0$ in V with $q(v) = 0$.

Ex: Hyperbolic Plane: $H = K^2$ with $G = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $q = 2x_1x_2$

equivalently $G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $q = x_1^2 - x_2^2$

Prop: Suppose (V, B) is non-degenerate. If v is an isotropic vector, then $V = H \hat{\oplus} U$ with $v \in H$.

Pf: $\exists\, z \in V$ with $B(v, z) = 1$. Set $w = z - \frac{1}{2}B(z,z)\,v$.

Then $B(w, v) = 1$ and $B(w,w) = B(z,z) + 2\left(-\frac{1}{2}B(z,z)\right) = 0$

So $\langle v, w \rangle \cong H$. Since (H, B) is non-degenerate,

$V = H \hat{\oplus} H^{\perp}$ by HW. ▨

Def: (V,B) is *isotropic* if it has an isotropic vector; otherwise it's *anisotropic*.

Cor: (V,B) non-singular. Then $V = H^n \oplus U$ with U $\underbrace{\text{anisotropic.}}_{\text{building blocks}}$

Q: What is $q(V)$? If (V,B) is non-singular and isotropic, then $q(V) = K$ (just look at H).

[Query: Example where $q(V) \neq K$; eg. $q = x_1^2 + x_2^2$ on $\mathbb{R}^2$.]

Hasse-Minkowski: q a quad. form on $\mathbb{Q}^n$. Then $\exists X \in \mathbb{Q}^n$ with $q(X) = 0$ iff for every place v there is a $X_v \in \mathbb{Q}_v^n$ with $q(X_v) = 0$.

i.e. a prime p or ∞.

[Also holds if we replace $\mathbb{Q}$ with a number field K.]
[Instead of "$\exists$ an isotropic vector" also say "q rep's 0".]

Ex: $q = x^2 - 7y^2$

$v = \infty$: reps 0, e.g. $(\sqrt{2}, 1)$

$v = 3$: reps 0 as $x^2 - 7y^2 = 0$ $\iff \left(\frac{x}{y}\right)^2 = 7 \iff 7$ is a square in $\mathbb{Q}_3$, and $1^2 \equiv 7 \bmod 3$.

$v = 5$: doesn't rep 0 as $z^2 \equiv 7 \bmod 5$ has no solutions.

For HM to be useful need better understanding of quad forms $/\mathbb{Q}_p$

Cor: $a \in \mathbb{Q}^\times$. q reps $a \iff q_v$ reps a for all v.

Pf: Apply HM to $a x_{n+1}^2 - q(x_1, \ldots, x_n)$, which reps 0 iff q reps a. $\blacksquare$

Thm: Two quad forms q_1 and q_2 on $\mathbb{Q}^n$ are isometric iff they are isometric over $\mathbb{Q}_v$ for all v.