Lecture 30: Quadratic Forms I

Throughout, K is a field of char $\neq 2$.

Quadratic Space: a finite dim'l vector space V over K with a symmetric bilinear form $B: V \times V \to K$.

i.e. $B(v,w) = B(w,v)$ and $B(kv_1+v_2, w) = kB(v_1,w) + B(v_2,w)$

Ex: ① $V = \mathbb{R}^n$ and $B(v,w) = \sum v_i w_i$

② $V = \mathbb{Q}^2$ $B(v,w) = 8v_1w_1 + v_1w_2 + v_2w_1 - 3v_2w_2$

Quadratic Form: $q: V \to K$ given by $q(v) = B(v,v)$

Note: q actually determines B via

$$B(v,w) = \frac{1}{2}\left(q(v+w) - q(v) - q(w)\right)$$

Ex: ① $q = v_1^2 + \cdots + v_n^2$ ② $q = 8v_1^2 + 2v_1v_2 - 3v_2^2$

Note: $q(v)$ can be negative, or even 0, e.g. in case ② $q((0,1)) = -3$ and $q((1,2)) = 8 + 4 - 3\cdot 2^2 = 0$.

B is determined by its Gramm matrix $G = (B(e_i,e_j))$ with respect to any basis $\mathcal{B} = \{e_1, \ldots, e_n\}$ of V:
In particular, if $x, y \in V$, then

column vector cor to w in basis $\mathcal{B}$

$$B(v,w) = [v]_{\mathcal{B}}^T G [w]_{\mathcal{B}} = (a_1, a_2, \dots a_n)\begin{pmatrix} g_{11} & g_{12} & \\ & \ddots & \\ & & \end{pmatrix}\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$= \sum_{i,j} a_i B(e_i, e_j) b_j = B(v,w) \quad \text{as } v = \sum a_i e_i, \; w = \sum b_i e_i$$

If $\mathcal{B}'$ is another basis of B, then

$$G_{\mathcal{B}'} = C^T G_{\mathcal{B}} C \quad \text{where } C = [Id]^{\text{Input } \mathcal{B}'}_{\text{Output } \mathcal{B}}$$

Discriminant: $\operatorname{disc}(q) = \det G_{\mathcal{B}}$, well-defined up to mult by $(K^\times)^2$.

Ex: ① $G = Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ so $\operatorname{disc} = 1$

② $G = \begin{pmatrix} 8 & 1 \\ 1 & -3 \end{pmatrix}$ so $\operatorname{disc} = -25 \sim -1 \mod (\mathbb{Q}^\times)^2$

While the only cond. on G that $x^T G y$ define a bilinear form is that $G^T = G$, we'll show:

Thm: (V, B) a quad. space. Then $\exists$ a basis $\mathcal{B}$ for V where $G_{\mathcal{B}}$ is diagonal, i.e. $q(v) = \sum a_i v_i^2$.

Ex ②: Take $e_1 = (0,1)$. If G is to be diagonal, then need $B(e_1, e_2) = 0 \iff w_1 - 3w_2 = 0$ where $e_2 = (w_1, w_2)$. So take $e_2 = (3,1)$. Then $G = \begin{pmatrix} -3 & 0 \\ 0 & 75 \end{pmatrix}$

(c.f. Gramm-Schmidt)

<u>Def</u>: For a subset W of V, let

$$W^{\perp} = \{ v \in V \mid B(v,w) = 0 \text{ for all } w \in W \}$$

If $v \in V$ has $B(v,v) \neq 0$, then $V = \langle v \rangle \oplus \{v\}^{\perp}$ since $\dim \{v\}^{\perp} \geq \dim V - 1$ as $\{v\}^{\perp} = \ker(\underbrace{v^T G}_{1 \times n \text{ matrix}})$

<u>Pf of Thm</u>: By induction, we find

$$V = \langle e_1 \rangle \mathbin{\hat{\oplus}} \langle e_2 \rangle \underset{\text{orthogonal direct sum}}{\hat{\oplus}} \langle e_3 \rangle \cdots \hat{\oplus} \langle e_k \rangle \mathbin{\hat{\oplus}} W \qquad \left(\begin{array}{cc|c} a_1 & 0 & 0 \\ 0 & a_k & \\ \hline & & 0 \ \ * \\ 0 & & * \ \ 0 \end{array}\right)$$

where $q(w) = 0$ for all $w \in W$. Since q determines B, it follows that $B(w_1, w_2) = 0$ for all $w_i \in W$. So completing $\{e_1, \dots, e_k\}$ to a basis of V using elts of W gives the needed basis. ▨

<u>Def</u>: B is <u>nonsingular</u> if every $v \neq 0$ in V has some $w \in W$ with $B(v,w) \neq 0$.

<u>Equivalent formulations</u>: ① $V^{\perp} = \{0\}$.

② $V \longrightarrow V^*$ is an isom
$v \longmapsto B(v, \cdot)$

③ $\operatorname{disc} = \det G \neq 0$.

Always have $V = V^{\perp} \oplus W$ where $B|_W$ is nonsingular

Prop: If B is non-singular, then for every subspace W
(i) $(W^\perp)^\perp = W$ (ii) $\dim W + \dim W^\perp = \dim V$

Note: Need not have $V = W \oplus W^\perp$ as sometimes
$W \cap W^\perp \neq \{0\}$, [e.g. if $w \neq 0$ in W with $q(w) = 0$.]

Pf: HW.

Classification of non-singular quad forms; up to isometry.

$\underline{K = \mathbb{C}}$: only one, namely example (1).

$\underline{K = \mathbb{R}}$: there are $\dim V$ of them, namely
$B_k = x_1^2 + \cdots + x_k^2 - (x_{k+1}^2 + \cdots + x_n^2)$. Signature $= \#\text{pos} - \#\text{neg}$
$= 2k - n$

Pf: Can choose a basis $\{e_1, e_2, \dots, e_n\}$ where
$G = \operatorname{diag}(d_1, \dots, d_n)$. Replacing e_i with λe_i changes
d_i to $\lambda^2 d_i$. ▨

Things are more complicated with $K = \mathbb{Q}$.

E.g. in example 2, $G = \begin{pmatrix} 8 & 1 \\ 1 & -3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -3 & 0 \\ 0 & 75 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -3 & 0 \\ 0 & 3 \end{pmatrix}$

replace e_2 by $\frac{1}{5} e_2$.

In fact, $G \rightsquigarrow \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

as if $C = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ then $C^t \begin{pmatrix} -3 & 0 \\ 0 & 3 \end{pmatrix} C = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

On the other hand $x_1^2 - 3x_2^2$ is really diff than $x_1^2 - x_2^2$
because of the disc...