Lecture 25: Units in integer rings III.

Dirichlet's Unit Thm: $\mathcal{O}_K^\times \cong \mu(K) \oplus \mathbb{Z}^{r+s-1}$

$$K^\times \xrightarrow{j} K_{\mathbb{R}}^\times \xrightarrow{l} \Big[\prod_{\tau} \mathbb{R}\Big]_F$$

$$\mathcal{O}_K^\times \longrightarrow \underset{\mathcal{N}=\pm 1}{S} \twoheadrightarrow \underset{tr=0}{H} \cong \mathbb{R}^{r+s}$$

Set $\Gamma = j(l(\mathcal{O}_K^\times))$. Then

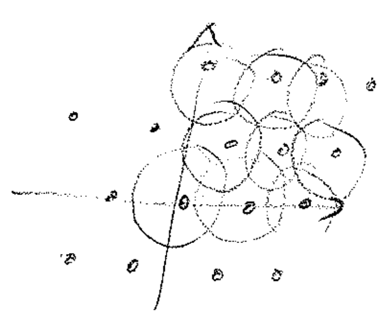
$$\left.\begin{array}{l} \underline{\text{Lemma 1}}:\ \mathcal{O}_K^\times / \mu(K) \cong \Gamma \ \checkmark \\ \underline{\text{Lemma 2}}:\ \Gamma \text{ is a complete lattice in } H. \end{array}\right\} \text{Imply the thm. } \checkmark$$

Proof of Lemma 2:

ⓐ Γ is discrete. ✓

ⓑ $H/_\Gamma$ is cpt: Equivalently, we need to find a compact set M in H where

$$H = \bigcup_{\gamma \in \Gamma} (\gamma + M) \quad \text{as then}$$



$M \twoheadrightarrow H/_\Gamma$ and so $H/_\Gamma$ is compact.

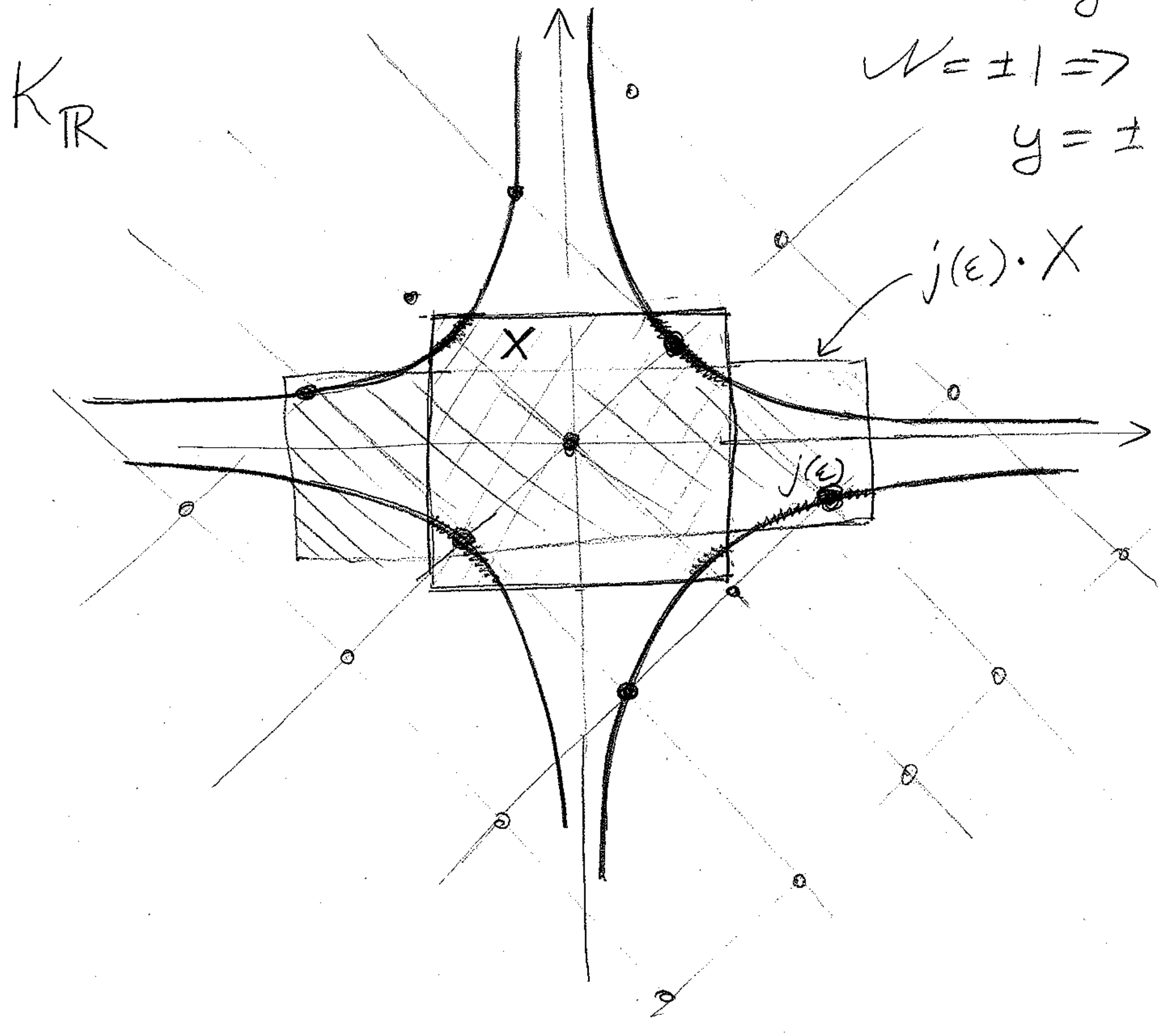
As $S \twoheadrightarrow H$, it suffices to find a compact set T in S so that $S = \bigcup_{\varepsilon \in \mathcal{O}_K^\times} j(\varepsilon)\cdot T$.

<u>Idea:</u> [Based on $K = \mathbb{Q}(\sqrt{2})$]

$\mathcal{N} = xy$

$\mathcal{N} = \pm 1 \Rightarrow$

$y = \pm \frac{1}{x}$



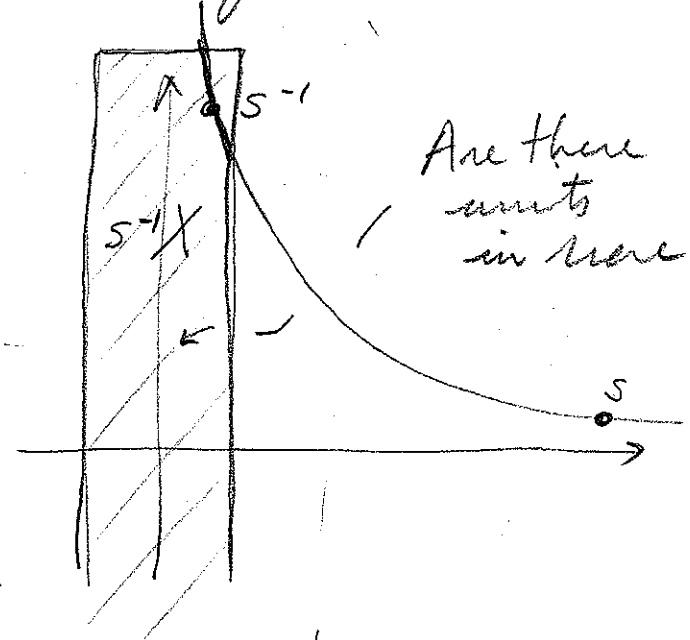
Let $X = \{\vec{x} \in K_{\mathbb{R}} \mid |x_\tau| < c\}$ [Pick c later.]

Do the $j(\varepsilon)\cdot X$ cover S? Consider some $s \in S$

Need $s \in j(\varepsilon)X \iff j(\varepsilon)^{-1} \in s^{-1}X$

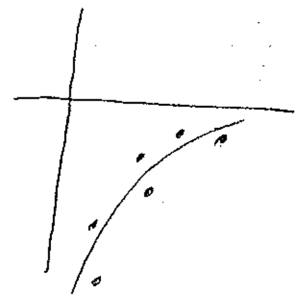
also in S

Units having inverses, this is equivalent to $j(\varepsilon') \in S^{-1}X$



Problem: We don't know anything about $j(\mathcal{O}_K^\times)$, but just $j(\mathcal{O}_K)$.

Key: Points of $j(\mathcal{O}_K)$ near S lie in finitely many $\mathcal{O}_K^\times$-orbits.



Actual Proof:

Choose c so that $c^n > \left(\frac{2}{\pi}\right)^s \sqrt{|\Delta_K|}$. As before

$$\mathrm{Vol}_{can}(X) > 2^n \, \mathrm{Vol}(j(\mathcal{O}_K))$$

and moreover $\mathrm{Vol}_{can}(sX) = \mathrm{Vol}_{can}(X)$ for all $s \in S$.

Claim: $\exists\, \alpha_1, \ldots, \alpha_n \in \mathcal{O}_K$ s.t. $\forall\, \alpha \in \mathcal{O}_K$ with

$$|N_{K/\mathbb{Q}}(\alpha)| < c^n \text{ then } \alpha = \varepsilon \alpha_i \text{ for some } \varepsilon \in \mathcal{O}_K^\times.$$

Deferring the claim, let's complete the proof of the theorem.

Set
$$T = S \cap \left(\bigcup j(\alpha_i)^{-1} X \right)$$
Want $S = \bigcup_{\varepsilon \in \mathcal{O}_K^\times} j(\varepsilon) T$. Now $s \in S$ is in some translate of $T \iff s \in j(\varepsilon \alpha_i)^{-1} X$
$\iff \varepsilon \alpha_i \in s^{-1} X \iff \alpha \in s^{-1} X$ with $|N_{K/\mathbb{Q}}(\alpha)| < C^n$

Given s, we know $s^{-1}X$ is large enough to contain some $\alpha \in j(\mathcal{O}_K)$, and $|N_{K/\mathbb{Q}}(\alpha)| = |N(j(\alpha))|$ $= |N(s\,j(\alpha))| < C^n$ as $s\,j(\alpha) \in X$. So we're done modulo the claim. $\blacksquare$

<u>Proof of Claim</u>: E.T.S. that there are only finitely many α with $N_{K/\mathbb{Q}}(\alpha) = a$ up to mult by units. The reason is that, up to mult by $\mathcal{O}_K^\times$ there is only one elt with norm a in each coset of $a\mathcal{O}_K$ in $\mathcal{O}_K$: If α and $\alpha + a\beta$ have norm a, then $\dfrac{\alpha + a\beta}{\alpha} = 1 + \underbrace{\left(\tfrac{a}{\alpha}\right)}_{\text{in } \mathcal{O}_K}\beta$, a unit. $\blacksquare$