Math 530: Problem Set 9

Due date: In class on Wednesday, April 22.
Course Web Page: `http://dunfield.info/530`

1. Here's an example where one doesn't have a local-to-global principle. Consider the equation $x^2 + 17y^2 = 257$. Show:

   (a) The equation has a solution in $\mathbb{F}_p$ for all p.

   (b) The equation has a solution in $\mathbb{Z}_p$ for all p.

   (c) Sadly, the equation has *no* solution in $\mathbb{Z}$.

   For (a) and (b), the harder cases are when $p \in \{2, 17, 257\}$. For part (b) with $p = 2$, one approach is to use Theorem 7.32 from page 114 of Milne's notes.

2. Suppose (V, B) is a quadratic space over a field K with $\mathrm{char}(K) \neq 2$, and that B is nondegenerate. Let W be a subspace of V. Show that:

   (a) $(W^\perp)^\perp = W$

   (b) $\dim W + \dim W^\perp = \dim V$.

   (c) Show that if (W, B) is nondegenerate, then $V = W \widehat{\oplus} W^\perp$, where $\widehat{\oplus}$ denotes *orthogonal* direct sum.

   (d) Give an example where (c) fails if (W, B) is degenerate. Here (V, B) should still be nondegenerate.

3. Let (V, B) be a 2-dimensional quadratic space over K. Prove that V is isotropic if and only if $-\mathrm{disc}(B)$ is a square in K.

4. Suppose a real symmetric 4×4 matrix G has characteristic polynomial $x^4 - dx^2 + 12$ for some $d \in \mathbb{R}$. Let B be the corresponding bilinear form on $\mathbb{R}$. Is B non-degenerate? Is it isotropic or anisotropic? What is its canonical form among those listed in class? (These things may depend on d, and note that not all d are possible.)

5. Let (V, B) be a nondegenerate quadratic space. Let x and y be anisotropic vectors in V with $q(x) = q(y)$. Show that there is an isometry τ of V such that $\tau(x) = y$.