Math 530: Problem Set 8[1]

Due date: In class on Wednesday, April 15.
Course Web Page: `http://dunfield.info/530`

1. Write $\frac{2}{3}$ and $-\frac{2}{3}$ as elements of $\mathbb{Z}_5$.

2. Using the definition of $\mathbb{Z}_p$ as the inverse limit of $\mathbb{Z}/p^k\mathbb{Z}$, prove that the ideals of $\mathbb{Z}_p$ are exactly the principal ideals $p^n\mathbb{Z}_p$ for $n \in \mathbb{Z}_{\geq 0}$. Thus $\mathbb{Z}_p$ is a PID.

3. In $\mathbb{Q}_5$, prove that
$$\sqrt{-1} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\binom{\frac{1}{2}}{n}5^n \quad \text{where} \quad \binom{r}{n} = \frac{r(r-1)(r-2)\cdots(r-n+1)}{n!} \quad \text{for any } r \in \mathbb{C}.$$

4. Recall a valuation on a field K is a function $|\cdot|\colon K \to \mathbb{R}_{\geq 0}$ where

   (a) $|x| = 0 \iff x = 0$
   (b) $|xy| = |x||y|$
   (c) $|x+y| \leq |x| + |y|$

   Note that $|\cdot|$ is always ≥ 0; I forgot to mention this when I defined valuations in class.

   A valuation is *archimedean* if $|n|$ is unbounded for $n \in \mathbb{N}$. Conversely, if $|n|$ is bounded it is *nonarchimdean.*

   Prove that $|\cdot|$ is nonarchimedean if and only if $|x+y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$.

5. Suppose $|\cdot|$ is a nonarchimedean valuation on a field K. The topology on K induced by $d(x,y) = |x-y|$ is decidedly odd, as you'll now demonstrate.

   (a) Denote the ball about $a \in K$ of radius $r \in \mathbb{R}_{>0}$ by
   $$B_r(a) = \{x \in K \mid |x-a| < r\}.$$
   Prove that if $b \in B_r(a)$ then $B_r(b) = B_r(a)$. Deduce that if two balls meet, then the larger radius one contains the smaller.

   (b) Prove that (K,d) is totally disconnected, that is every open set is the *disjoint* union of two nonempty open sets.

   (c) Assume (K,d) is a complete metric space, i.e. every Cauchy sequence converges. Prove that for $a_n \in K$, the series $\sum a_n$ converges if and only if $a_n \to 0$.[2]

6. Show that $5x^3 - 7x^2 + 3x + 6$ has a root $\alpha \in \mathbb{Z}_7$ with $|\alpha - 1|_7 < 1$. Find $a \in \mathbb{Z}$ such that $|\alpha - a|_7 \leq 7^{-4}$.

7. Prove that $\mathbb{Q}_p$ contains the $(p-1)^{\text{st}}$ roots of unity. Does it contain any other roots of unity?

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[1] Revision of April 15, 2009; Changes: Clarified problem 4.
[2] If only this were true for $K = \mathbb{R}$, then teaching Calc II would be much easier...