Math 530: Problem Set 7

Due date: In class on Wednesday, April 8.
Course Web Page: `http://dunfield.info/530`

1. In this problem, you'll consider the connection between the discriminant and the degree of a number field.

   (a) Show that the only number field K with $|\Delta_K| = 1$ is $\mathbb{Q}$. An immediate consequence is that for any number field $K \neq \mathbb{Q}$ there is some rational prime that ramifies in $\mathcal{O}_K$.

   (b) Show that $|\Delta_K| \to \infty$ as $[K : \mathbb{Q}] \to \infty$.

2. Let K be a number field and $\mathfrak{a} \subset \mathcal{O}_K$ an ideal.

   (a) Prove there exists a finite extension L of K so that $\mathfrak{a}\mathcal{O}_L$ is principle.

   (b) Does there always exists a finite extension L in which *every* ideal of $\mathcal{O}_K$ becomes principle in $\mathcal{O}_L$? Prove your answer.

3. For a number field K, the order of the ideal class group is called the *class number* and usually denoted h. Show that the quadratic fields with discriminant $5, 8, 12, -3, -4, -7, -8, -11$ have class number 1.

4. Show that the class groups of $\mathbb{Q}(\sqrt{10})$ and $\mathbb{Q}(\sqrt{-10})$ are both $\mathbb{Z}/2\mathbb{Z}$.

5. Compute the class group of $\mathbb{Q}(\sqrt[3]{7})$.

6. Let K be a totally real number field, i.e. the image of every embedding $K \to \mathbb{C}$ is contained in $\mathbb{R}$. Let T be a proper nonempty subset of the set of embeddings $\tau\colon K \to \mathbb{R}$. Prove there exists a unit $\epsilon \in \mathcal{O}_K^\times$ satisfying $0 < \tau(\epsilon) < 1$ for $\tau \in T$ and $\tau(\epsilon) > 1$ for $\tau \notin T$.

   Hint: Apply Minowski's lattice point theorem to the image of the unit lattice in the trace-zero subspace.