Math 530: Final Problem Set.[1]

All quaternions, all the time.

Due date: In class on Wednesday, May 6.
Reminder: Our final will be on Thursday, May 14 from 8-11am in our usual classroom.

1. Let K be a field of characteristic $\neq 2$, and chose $a, b \in K^\times$. Consider the following associative algebra over K: let A be the K vector space with basis $\{1, i, j, k\}$ and multiplication determined by $i^2 = a$, $j^2 = b$, and $ij = -ji = k$. The algebra A is called a *quaternion algebra*, and these are very important examples in number theory. The algebra A is sometimes denoted by its *Hilbert symbol* $\left(\frac{a,b}{K}\right)$. For instance, Hamilton's original, accept no substitutes, quaternions are $\mathscr{H} = \left(\frac{-1,-1}{\mathbb{R}}\right)$.

   (a) Prove that $M_2(K)$, the algebra of 2×2 matrices is a quaternion algebra. Hint: It's $\left(\frac{1,1}{K}\right)$.

   (b) Different Hilbert symbols can give rise to isomorphic quaternion algebras. Give an example.

   (c) Prove that the only quaternion algebra over $\mathbb{C}$ is $M_2(\mathbb{C})$ and the only two over $\mathbb{R}$ are $M_2(\mathbb{R})$ and $\mathscr{H}$.

2. Let $A = \left(\frac{a,b}{K}\right)$. For $\alpha = w + xi + yj + zk$, define its conjugate to be $\overline{\alpha} = w - xi - yj - zk$. Then we can define the *norm* $\mathscr{N}\colon A \to K$ by $\mathscr{N}(\alpha) = \alpha\overline{\alpha}$ and *trace* $\operatorname{tr}\colon A \to K$ by $\alpha + \overline{\alpha}$.

   (a) Calculate the norm and trace explicitly for $\mathscr{H}$.

   (b) Show that $\mathscr{N}$ gives a quadratic form on A, which is diagonal with respect to the standard basis $\{1, i, j, k\}$.

   (c) Show that the norm and trace are multiplicative and additive, respectively. Hint: There's a nice formula for $\overline{\alpha\beta}$, which typically isn't equal to $\overline{\alpha}\,\overline{\beta}$.

   (d) What standard quantities are the norm and trace on $M_2(K)$?

   (e) The subspace $A_0 = \{\alpha \in A \mid \operatorname{tr}(\alpha) = 0\}$ is called the *pure quaternions*. For Hamilton's quaternions, we have $\mathscr{H}_0 \cong \mathbb{R}^3$. Prove that quaternion multiplication on $\mathscr{H}_0$ is a combination of the usual dot and cross products as follows: $\alpha\beta = \alpha \times \beta - \alpha \cdot \beta$. This is the source of the convention in vector calculus that the standard basis of $\mathbb{R}^3$ is $\{i, j, k\}$.

3. Recall that an algebra is a *division algebra* if every nonzero element has a multiplicative inverse[2]. Let $A = \left(\frac{a,b}{K}\right)$. It is not hard to show that A is a central simple algebra over K; thus by Wedderburn's theorem it is either $M_2(K)$ or a division algebra. Prove that the following are equivalent:

   (a) $A \cong M_2(K)$; equivalently, A is not a division algebra.

   (b) The norm form on A has an isotropic vector.

   (c) The norm form on A_0 has an isotropic vector.

   (d) The usual Hilbert symbol $(a, b) = 1$.

[1] Revised May 1, 2009 to fix problem 2(e).

[2] A synonym for division algebra is "noncommutative field" which is a good way to think about such objects.