Lecture 15: More on covers.

Last time:

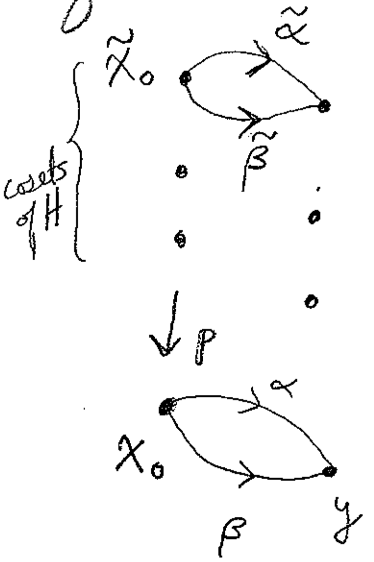
Thm: X path conn., locally path conn, semi-locally simply connected. Then X has a simply conn. cover.

Proof was very formal: $\tilde{X} = \{[\alpha] \mid \alpha \text{ a path from } x_0\}$

$$\begin{array}{ccc} \tilde{X} & & [\alpha] \\ \downarrow p & & \downarrow \\ X & & \alpha(1) \end{array}$$

Thm: X as above. Given $H \le \pi_1(X, x_0)$, $\exists$ a covering $p: (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ s.t. $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = H$.

Pf. Take $\tilde{X} = \{[\alpha] \mid \alpha \text{ a path } x_0\} / \sim$



$\tilde{x}_0 = [\text{const}_{x_0}]$

where $[\alpha] \sim [\beta] \iff [\alpha \cdot \bar{\beta}] \in H$

Same p and topology as before

$p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ = loops lifting to loops at $\tilde{x}_0$ = $[\alpha] \sim [\text{const}_{x_0}]$ = $[\alpha] \in H$.

Idea: α and β label the same pt above y if $\alpha \cdot \bar{\beta}$ lifts to a loop at $\tilde{x}_0$, i.e. $[\alpha \cdot \bar{\beta}] \in H$. ∎

[Foreshadow meaning of "universal cover".]

<u>Cor</u>: Let F be a free group. Then any subgroup of F is also free.

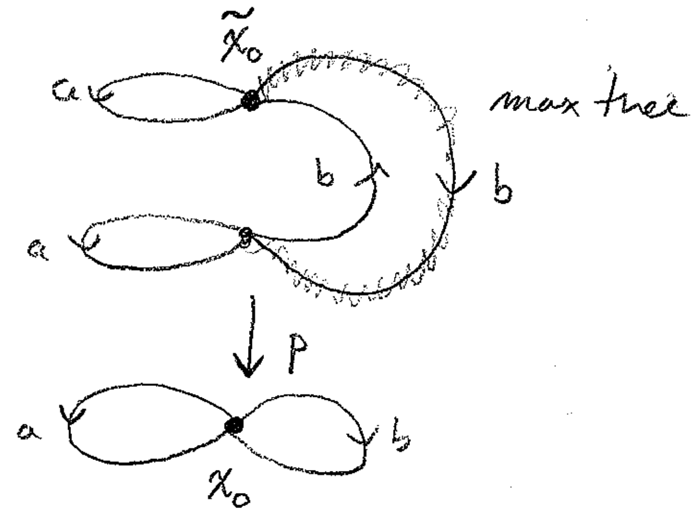
<u>Proof</u>: $F = \pi_1(X = \bigvee_\alpha S^1)$. Given $H \leq F$, let $\tilde{X}$ be the corresponding cover.

<u>Claim</u>: $\tilde{X}$ is also a graph

If so, we're done as $H \cong \pi_1 \tilde{X} =$ free group.

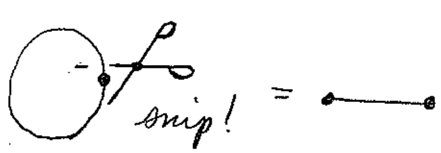
$\tilde{X}^0 = p^{-1}(X^0)$ and X^1 adds paths coming from lifts of edges. ▨

<u>Ex</u>: F_2, $H = p_*(\tilde{X})$ is free on a, b^2, bab^{-1}

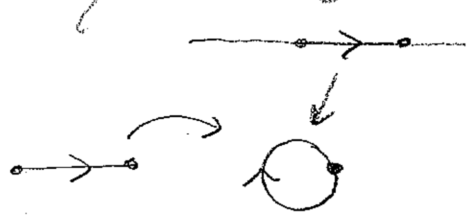


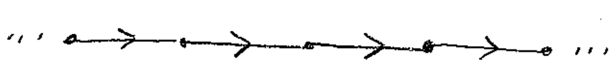
[On HW, you'll refine this.]

<u>Building the universal cover</u>: [Cut sp into simply conn pieces, assemble in a simply conn way.]

<u>Ex</u>: 

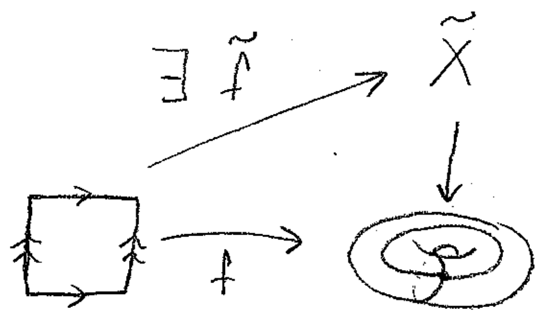
By path lifting, the interval lifts to any cover:

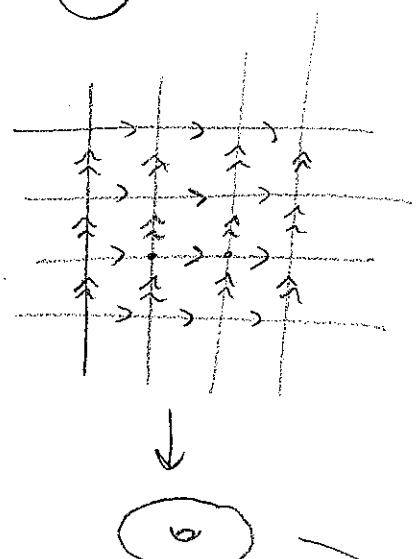


$\tilde{X} =$ 

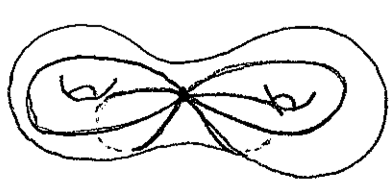
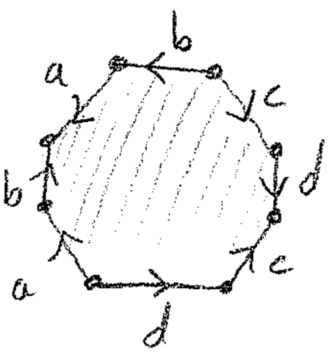
$= \mathbb{R}$.

<u>Ex</u>: 





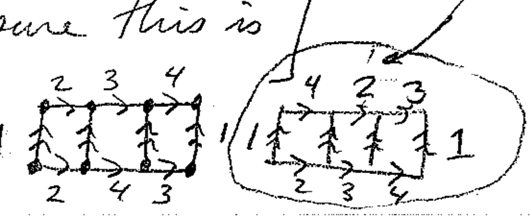
So $\tilde{X} = \mathbb{R}^2$ (also see from prod. str.)

<u>Ex</u>:  = 

Handout copy of tiling of hyp. plane.

[For the cor. question about building the cover cor. to H, see HW.]

<u>Q</u>! [What do I need to check to make sure this is a cover?]

One of these is a cover, which? 

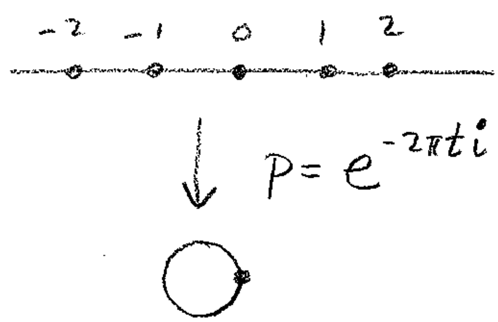
<u>Def</u>: Covers $\tilde{X}_1$ and $\tilde{X}_2$ of X are <u>isomorphic</u> if $\exists$ a homeo $f: \tilde{X}_1 \longrightarrow \tilde{X}_2$ such that $p_2 \circ f = p_1$.

$$\begin{tikzcd} \tilde{X}_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & \tilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

<u>Note</u>: <u>not</u> enough that $\tilde{X}_1 \cong \tilde{X}_2$, c.f. the covers $S^1 \to S^1$ given by $z \to z^n$.

When $\tilde{X}_1 = \tilde{X}_2$, such isoms are called <u>covering transformations</u>.

<u>Ex</u>:



<u>Cover trans</u>:
Fix k in $\mathbb{Z}$.
$f(t) = t + k$

<u>Fact</u>: These are all the covering trans.

<u>Notice</u>: $S^1 = \mathbb{R} / (\text{covering trans} \cong \pi_1 S^1)$

<u>Sim</u>:  $= \mathbb{R}^2 / ($cover trans $=$ trans by elts of $\mathbb{Z}^2 \cong \pi_1($  $))$

<u>Also</u>:

 , ...

<u>Thm</u>: X path conn, loc. path conn.

Then 2 path conn covering spaces are equiv via an isom f with $f(\tilde{x}_1) = \tilde{x}_2$

$\Longleftrightarrow$ $p_*(\pi_1(\tilde{X}_1, \tilde{x}_1)) = p_*(\pi_1(\tilde{X}_2, \tilde{x}_2))$

$$\begin{tikzcd}
(\tilde{X}_1, \tilde{x}_1) \arrow[rr, dashed, "f"] \arrow[dr, "p_1"'] & & (\tilde{X}_2, \tilde{x}_2) \arrow[dl, "p_2"] \\
& (X, x_0) &
\end{tikzcd}$$

Gives uniqueness to the earlier theorem.

Will prove after solving the "lifting problem".