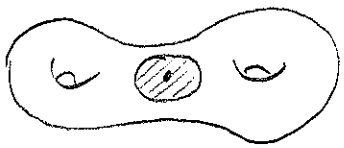
Lecture 14: Universal Covers

__Goal:__ X "reasonable". Given $H \leq \pi_1 X$, then $\exists !$ cover $p: \tilde{X} \to X$ with $p_*(\pi_1 \tilde{X}) = H$.

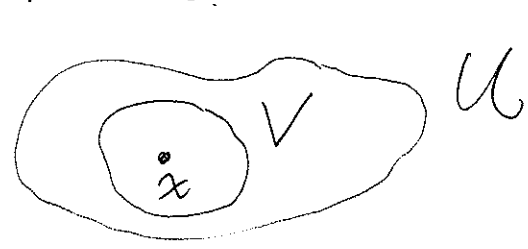
__Today:__ When does X have universal cover, i.e. one where $\tilde{X}$ is simply connected? Does  have such a cover?

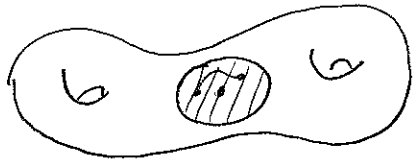
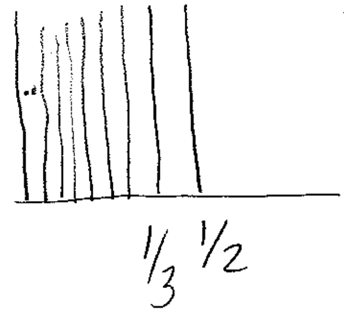
__Last time:__ If X has a univ cover, then every pt has a nbhd U with $i_*(\pi_1 U) = 1$ in $\pi_1 X$.

__Def:__ X is __semi-locally simply connected__ (S.L.S.C.) if every pt has a nbhd U with $i_*(\pi_1 U) = 1$ in $\pi_1 X$.

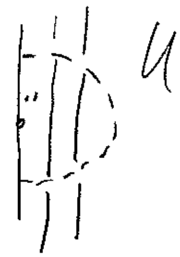
__Ex:__  __Non-Ex:__ 

__Def:__ A space X is __locally path connected__ if for each nbhd U of a pt x, $\exists$ a __path connected__ nbhd V with $x \in V \subseteq U$.



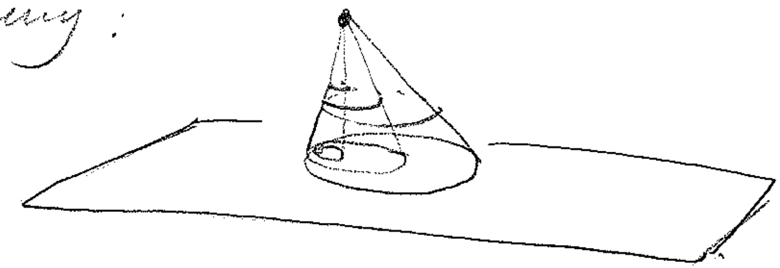
Ex:  Non Ex: 

path conn, but not loc. path conn.



Local. simp. conn:
Replace "path conn" with simp. conn in the above.

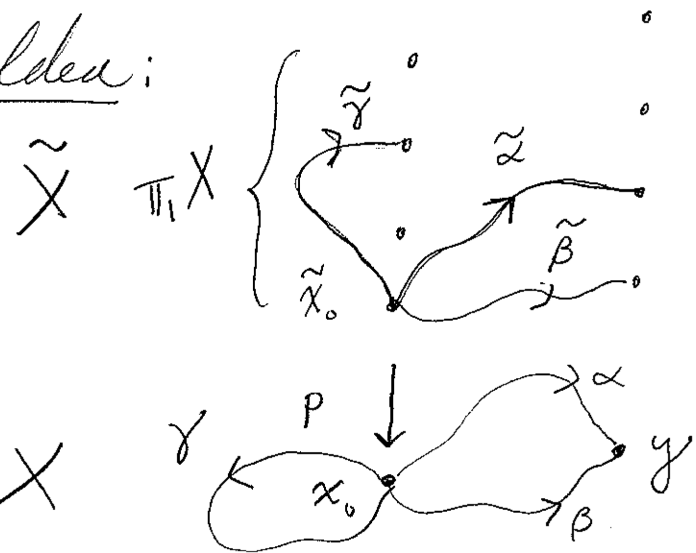
S.L.S.C but not L.S.C:
Cone the Hawaiian Earring:



Thm: X path conn, loc. path conn, S.L.S.C.
Then X has a univ. cover.

[Think path conn + loc. simp. conn $\Rightarrow$ univ cover
E.g.  has a univ cover. Q: what is it?]

Idea:



If $\tilde{\alpha}(1) = \tilde{\beta}(1)$, then $\alpha \cdot \beta^{-1} \in p_*(\pi_1 \tilde{X}) = 1$ $\Rightarrow \alpha \simeq_p \beta$. Converse also true, so can ident $p^{-1}(y)$ with homotopy classes of paths from x_0 to y.

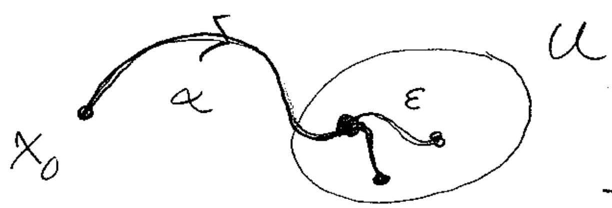
[works even when $y = x_0$.]

<u>Pf sketch</u>:

$$\begin{array}{l} \tilde{X} = \left\{ [\alpha] \;\middle|\; \begin{array}{l}\alpha \text{ a path} \\ \text{starting at } x_0\end{array} \right\} \\ \downarrow p \\ X \qquad \text{where } p([\alpha]) = \alpha(1). \end{array}$$

The top on X has a basis $\mathcal{B}$ where each open set U is path conn and $\pi_1 U \to \pi_1 X$ is trivial. For each $U \in \mathcal{B}$ define

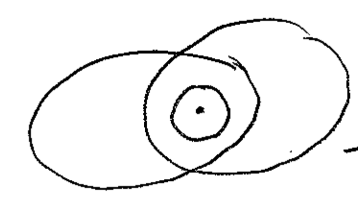
$$U_{[\alpha]} = \left\{ [\alpha \cdot \varepsilon] \;\middle|\; \begin{array}{l}\varepsilon \text{ a path} \\ \text{contained in } U\end{array} \right\}$$



<u>Note</u>: $p : U_{[\alpha]} \to U$ is a bijection.

Give $\tilde{X}$ the topology with basis

$$\{ U_{[\alpha]} \mid U \in \mathcal{B}, [\alpha] \in \tilde{X} \}$$

[Need to check this is really a basis ]

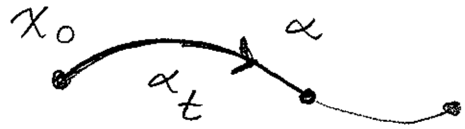
For $U \in \mathcal{B}$, then $p^{-1}(U) = \coprod\limits_{\substack{[\alpha] \text{ a} \\ \text{path from} \\ x_0 \text{ to } y_0 \\ \text{in } U.}} U_{[\alpha]}$

Hence p is cont.

Moreover,

$p|_{U_{[\alpha]}} : U_{[\alpha]} \to U$ is a homeo, so p is a covering map.

<u>Finially:</u>



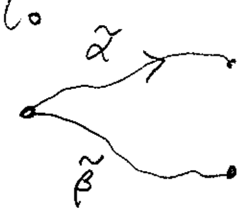
① $\tilde{X}$ is path conn: $[\alpha_t]$ gives a path from $[\text{const}_{x_0}]$ to $[\alpha]$

② $\pi_1 \tilde{X} \cong$ subgp of $\pi_1 X$ of loops lifting to loops in $\tilde{X}$. $= 1$.

$\blacksquare$

<u>Thm</u>: X as before, $H \le \pi_1(X, x_0)$. Then $\exists\, p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ with $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = H$.

<u>Pf</u>: Take $\tilde{X} = \{[\alpha] \mid \alpha \text{ a path from } x_0\} \big/ \sim$



$[\alpha] \sim [\beta]$ if $[\alpha \cdot \bar{\beta}] \in H$.

$\tilde{x}_0 = [\text{const}_{x_0}]$

$\blacksquare$