Lecture 11: π_1 of CW complexes

Last time:

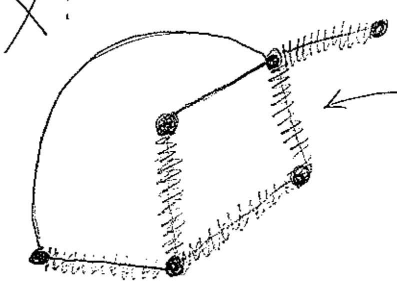
Thm: (X, A) has the homotopy ext. prop.
If A is contractible, then $X \simeq X/A$.

Ex: X a CW complex, A a subcomplex has the hom. ext. property.

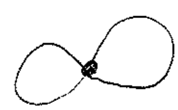
Thm: X a connected graph. Then $\pi_1 X$ is a free group.

Idea: Know $\pi_1 (\bigvee_\alpha S^1 =$  $)$ = free group on the loops.

X:



A = maximal tree, containing every vertex. Contractible

$\Longrightarrow X \simeq X/A =$ 

Need to show:

Every connected graph contains a tree passing through every vertex. Argue first for finite graphs.

graph w/o loops.

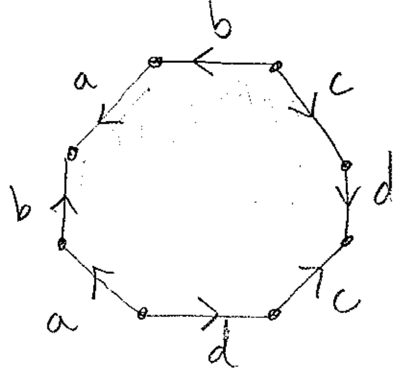
$\mathcal{T} = \{$tree subgraphs of $X\}$, partially ordered by $\subseteq$

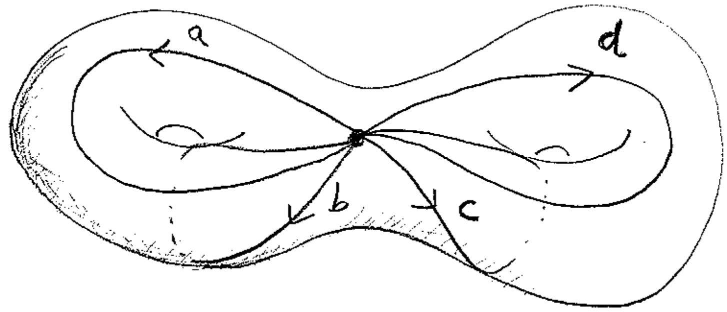
Any chain (tot. ordered subset) has an upper bound namely the union. By Zorn's Lemma, $\exists$ a max elt of $\mathcal{T}$, which must contain every vertex for the same reason as in the finite case. ▨

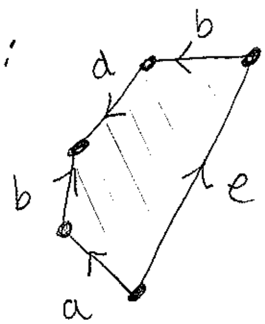
[So far can compute π_1 of things $\simeq$ to graphs and products of such.]

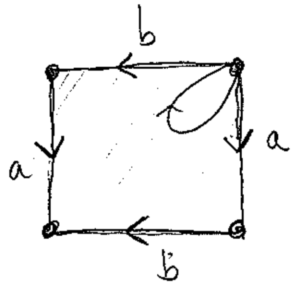
<u>What is π_1</u> of <u>X</u> =  ?

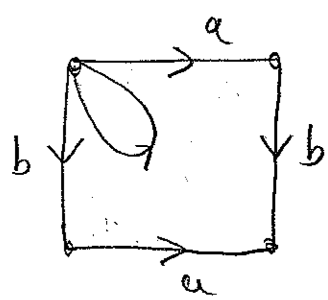
<u>Claim:</u> X is a CW complex.

X = 

= 

<u>Reason:</u> 

= 

= 

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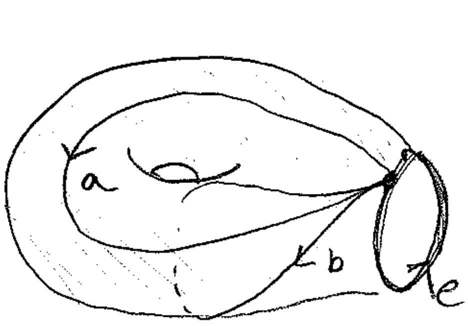
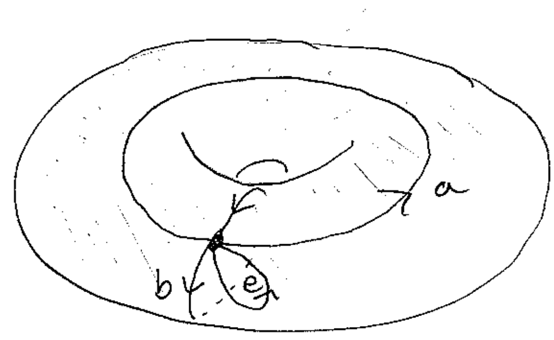
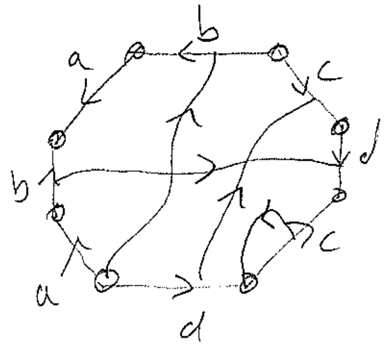


= 

Take any elt of $\pi_1(X, \text{vertex})$. Can rep by a piecewise-linear path.



Picking a pt not in path can push onto X'!

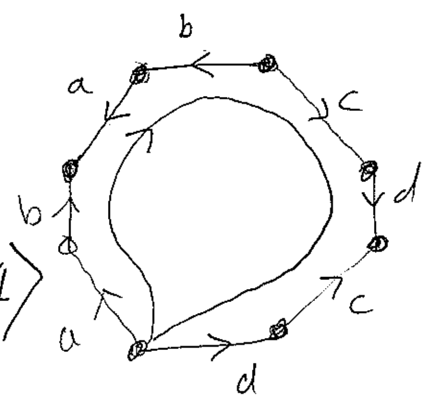
Thus: $\pi_1 X$ is gen by image of $\pi_1 X'$ under inclusion.

[True for all CW-cplxs]

Q: Could $\pi_1 X = \pi_1 X' = \text{Free Group}(a,b,c,d)$?

A: No, as $aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1$ in $\pi_1 X$.

Fact:



$$\pi_1 X = \langle a,b,c,d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1 \rangle$$

$$\underset{def}{=} \text{Free Group}(a,b,c,d) \Big/ \underbrace{\text{Normal closure of } w = aba^{-1}b^{-1}cdc^{-1}d^{-1}}_{\substack{\text{smallest normal subgp containing } w. \\ = \text{prod of conj of } w.}} = \langle w \rangle$$

= "Largest gp sat the relation w."

Ex: $\langle a, b \mid aba^{-1}b^{-1} = 1 \rangle = \mathbb{Z}^2$

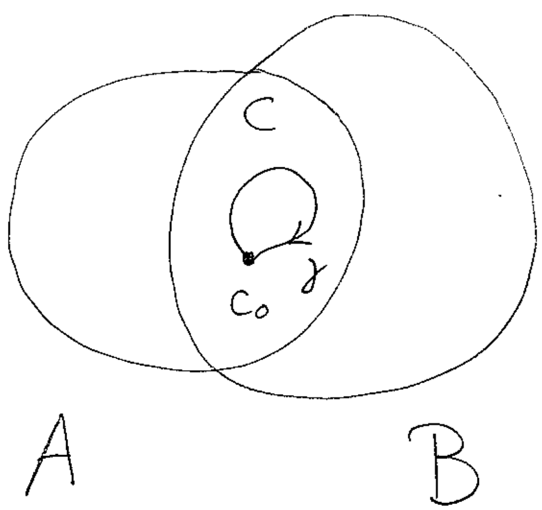
$\langle a \mid a^n = 1 \rangle = \mathbb{Z}^n$ $\quad \langle a, b \mid a^n = b^2 = \underbrace{(ab)^2}_{} = 1 \rangle = D_{2n}$

$abab = 1 \Leftrightarrow$

$bab^{-1} = a^{-1}$

Van Kampen's Thm: Suppose $X = A \cup B$ where $A, B, C = A \cap B$ are path connected open sets. For $c_0 \in C$ we have

$$\pi_1(X, c_0) = \pi_1(A, c_0) * \pi_1(B, c_0) \Big/ \left\langle \left\{ i_{A_*}(\gamma) \cdot i_{B_*}(\gamma)^{-1} \,\middle|\, \gamma \in \pi_1 C \right\} \right\rangle$$



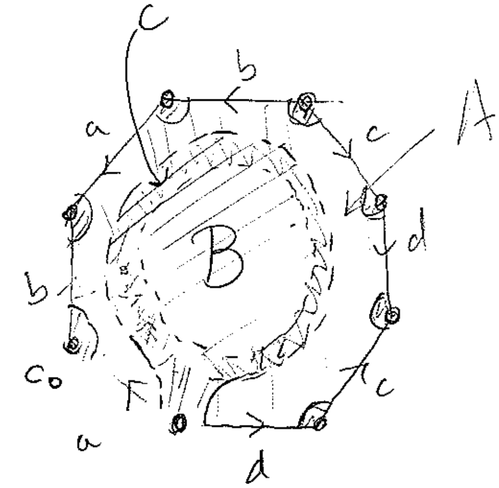
where

$i_A : A \hookrightarrow X$

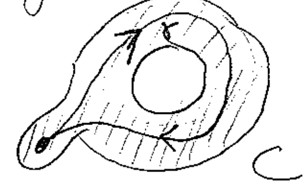
$i_B : B \hookrightarrow X$

are the inclusions

Ex: For X, take A = nbhd of X', B = disc shown



$\pi_1 C = \mathbb{Z}$ gen by γ.



$i_B(\gamma) = 1$

$i_A(\gamma) = aba^{-1}b^{-1}cdc^{-1}d^{-1}$

$$\left(\pi_1 A * \underbrace{\pi_1 B}_{=1} = \pi_1 A\right) \Big/ \langle i_A(\gamma) \cdot i_B(\gamma)^{-1} \rangle$$

$$= \text{Free Group}(a,b,c,d) \Big/ \langle aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1 \rangle$$

$$= \langle a,b,c,d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1 \rangle.$$

If time remains, define free product of groups.