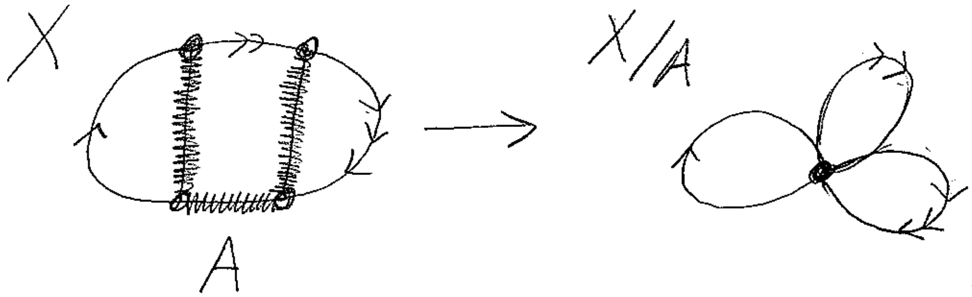
Lecture 10: Crushing contractible subspaces.

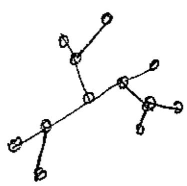
<u>Motivation</u>: To compute π_1(Graph), want that



is a homotopy equivalence.

<u>Point</u>: A doesn't much topology, in the following sense:

<u>Def</u>: A space A is contractible if it is homotopy equivalent to a pt. Equiv, $id_A \simeq (\text{Const map } A \mapsto a_0)$

<u>Ex</u>: $\mathbb{R}^n$, a tree 

<u>Non Ex</u>: Any $S^n \subseteq \mathbb{R}^{n+1}$, even though $\pi_1 = 1$ when $n > 1$. [Can't see with pres. technology.]

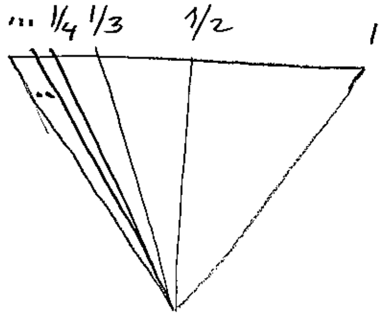
Suppose $A \subseteq X$ is a $\underbrace{\text{closed}}_{\text{just so } X/A \text{ is Hausdorff (HW.)}}$, contractible set.

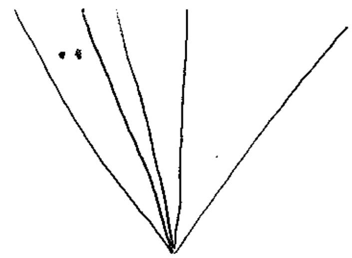
<u>Q</u>: Is $X \simeq X/A$?

has quotient top from last time

<u>A</u>. Not always.

$X \subseteq \mathbb{R}^2$: $\qquad\qquad$ A:





<u>Query</u>: What is X/A? [We've seen it before!]

<u>Ans</u>: Hawaiian earing . So $X/A \not\simeq X$ as $\pi_1 X/A$ is uncountable, but $\pi_1 X$ is countable, gen by 

and . [Explain only very roughly]

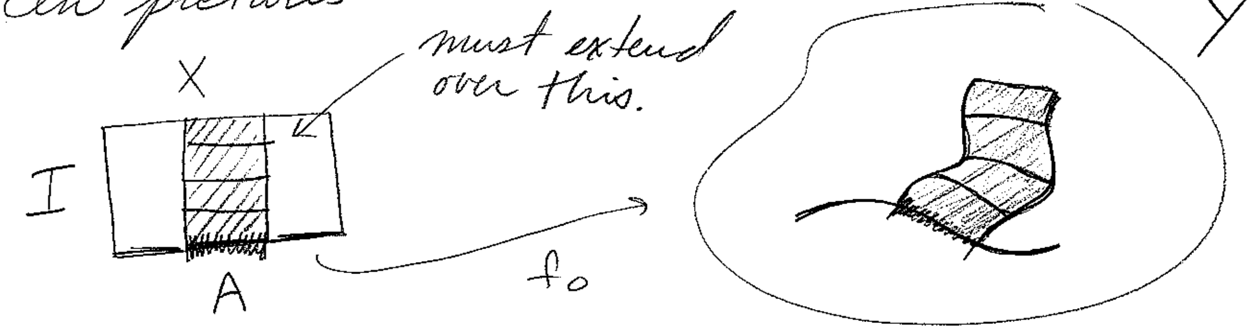
<u>Thm</u>: Suppose X is a CW complex, $\underbrace{A \text{ a subcomplex}}$.
If A is contractible, then $X \to X/A$ is a homotopy equiv.

= a union of cells, itself a CW complex.

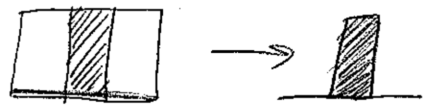
[This applies to the sit. of graphs we started with.]

<u>Def</u>: (X,A) have the <u>homotopy</u> <u>extension</u> <u>property</u> is given $f_0 : X \to Y$ and a homotopy $f_t : A \times I \to Y$ of $f_0|_A$, then f_t extends to a homotopy over all of X of f_0.

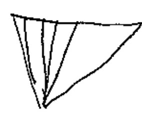
In pictures



<u>Easy fact</u>: This is equiv to $X \times I$ retracting to $X \times \{0\} \cup A \times I$.



<u>Ex</u>: $[0,1] \subseteq [-1,2]$, $A \subseteq X$ a subcplx of a CW cplx,

<u>Non ex</u>:  example above, also $X = I$, $A = \{1/n \mid n \in \mathbb{N}\}$

<u>Thm</u>: Suppose (X,A) has the hom. ext. prop. If A is contractible, then $X \xrightarrow[q]{} X/A$ is a $\simeq$.

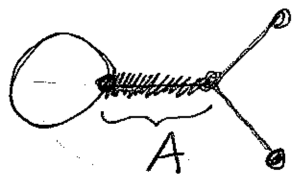
<u>Pf</u>: $\exists\, f_t : X \to X$ w/ $f_0 = id_X$, $f_t(A) \subseteq A$ and $f_1(A) = pt$. Have

$$\begin{tikzcd} X \arrow[r, "f_t"] \arrow[d, "q"'] & X \arrow[d, "q"] \\ X/A \arrow[r, "\bar{f}_t"] & X/A \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f_1"] \arrow[d, "q"'] & X \arrow[d, "q"] \\ X/A \arrow[r, "\bar{f}_1"] \arrow[ur, "g"] & X/A \end{tikzcd}$$

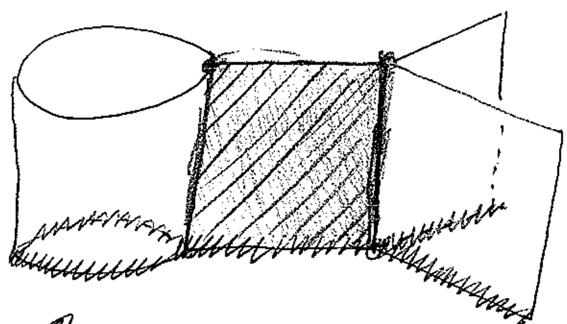
when g is the map induced by f_1

<u>Note</u>: q and g are inverse homotopy equivs as $g \circ q = f_1 \simeq id_X$ and $q \circ g = \bar{f}_1 \simeq id_{X/A}$.

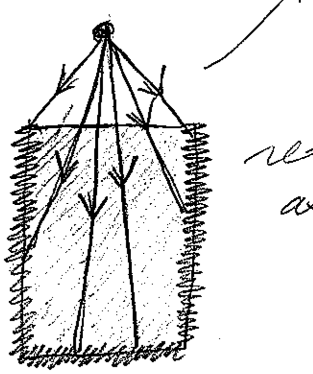
Thm: X a graph, A a subgraph. Then (X,A) has the hom. ext. prop.

Pf by example: $X =$ 

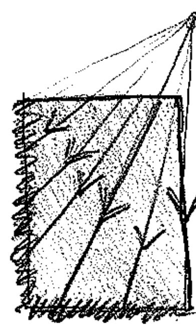
$X \times I$



Need to retract $X \times I$ to $X \times \{0\} \cup A \times I$



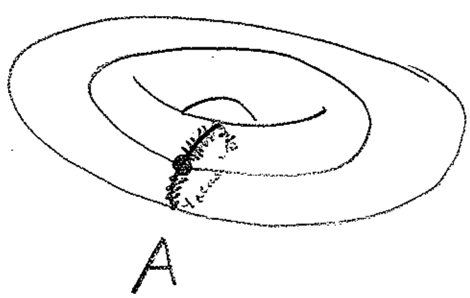
retract as shown

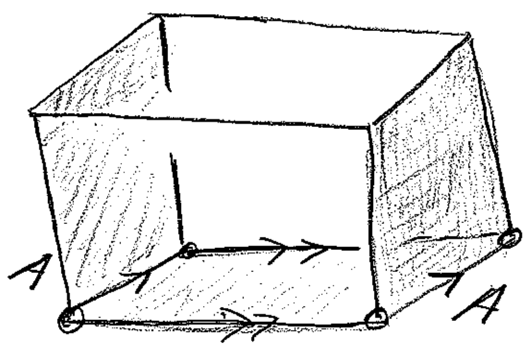


retract as shown

This gen to the case when X is a CW complex and A a subcomplex. **Key:** induction

X





$X \times I$.

See pgs 14-17 for more apps, including

Thm: (X,A) has the hom ext. prop. If $A \xrightarrow[i]{} X$ is a $\simeq$, then X def retracts to A.

If time remains, discuss why every conn. graph has a max tree which contains every vertex.