Lecture 17: Classifying Covering Spaces

Last time:

Thm: Covering spaces of a path conn, loc. path conn space X are isomorphic via an f taking $\tilde{x}_1$ to $\tilde{x}_2$ iff $p_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1)) = (p_2)_*(\pi_1(\tilde{X}_2, \tilde{x}_2))$

$$\begin{tikzcd}
(\tilde{X}_1, \tilde{x}_1) \arrow[rr, dashed, "f"] \arrow[dr, "p_1"'] & & (\tilde{X}_2, \tilde{x}_2) \arrow[dl, "p_2"] \\
& (X, x_0) &
\end{tikzcd}$$

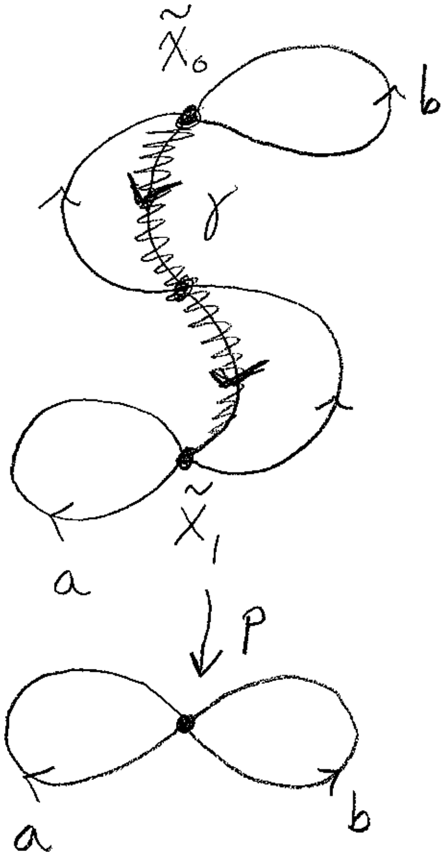
Mention survey results

Thm: X path conn, loc path conn, S.L.S.C.

$$\left\{\begin{array}{l}\text{isom classes of}\\ \text{path conn}\\ \text{covers } p:(\tilde{X},\tilde{x}_0)\to(X,x_0)\end{array}\right\} \xleftrightarrow{\text{bijection}} \begin{array}{l}\text{Subgroups}\\ \text{of } \pi_1(X,x_0)\end{array}$$

Note: The L.H.S. is classes of covers w/ choice of $\tilde{x}_0$ in $p^{-1}(x_0)$

Ex:



$p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \ni b$ but not a

$p_*(\pi_1(\tilde{X}, \tilde{x}_1)) \ni a$ but not b

Notice that $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ does contain $p_*(\gamma \cdot a \cdot \bar{\gamma}) = a b a b^{-1} a^{-1}$, and in fact

$$p_*(\gamma)\, p_*(\pi_1(\tilde{X}, \tilde{x}_1))\, p_*(\gamma)^{-1} = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$$

Thm: X as above, then

$$\left\{ \begin{array}{c} \text{isom class of} \\ \text{path conn} \\ \text{covers } \tilde{X} \to X \end{array} \right\} \xleftrightarrow{\text{bijection}} \left\{ \begin{array}{c} \text{conj classes} \\ \text{of subgps} \\ \text{of } \pi_1 X \end{array} \right\}$$

Application: Subgps of free gps are free

(Query class about this.)

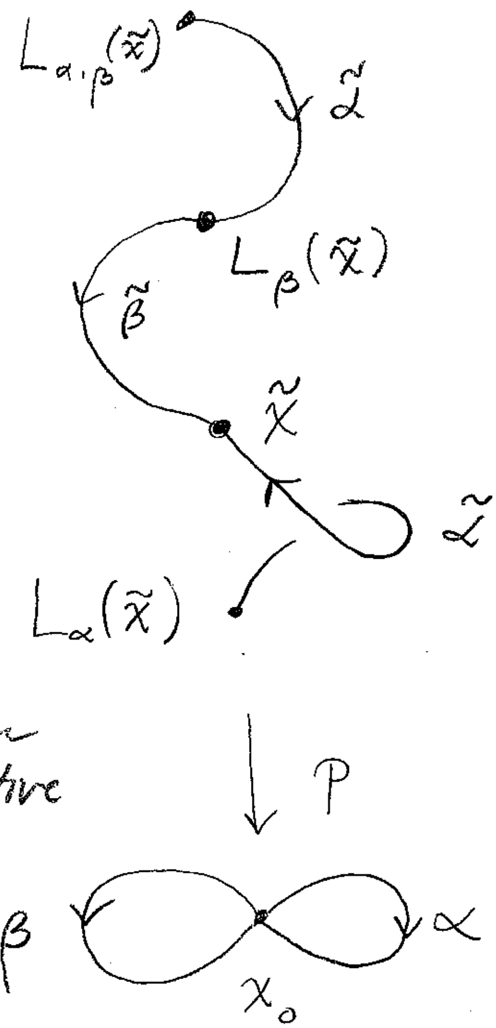
Action on fibers:

$p : \tilde{X} \longrightarrow X$ a cover, $\alpha \in \pi_1(X, x_0)$.

Define $L_\alpha \in Sym(p^{-1}(x_0))$ by

$$L_\alpha(\tilde{x}) = \tilde{\alpha}(0) \text{ where } \tilde{\alpha} \text{ is the lift of } \alpha \text{ \underline{ending} at } \tilde{x}.$$

<u>Key</u>: $L_{\alpha \cdot \beta} = L_\alpha \circ L_\beta$



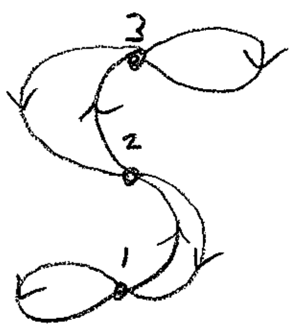
$\Rightarrow$ L_α is really a bijection and $L : \pi_1(X, x_0) \longrightarrow Sym(p^{-1}(x_0))$ is a homomorphism.

skip to example

[Query:] $Stab(\tilde{x}) = p_*(\pi_1(\tilde{X}, \tilde{x}))$

[Query:] $\tilde{X}$ is path conn $\Longleftrightarrow$ the action is transitive

<u>Ex</u>:



$L_a(1) = 1$

$L_a(2) = 3$

$L_a(3) = 2$

$\pi_1(\infty) \longrightarrow S_3$

$a \longmapsto (23)$

$b \longmapsto (12)$

<u>Thm</u>: X path conn, loc. path conn, S.L.S.C.

$$\left\{\begin{matrix}\text{conn. covers of} \\ X \text{ with } n\text{-sheets} \\ \text{w/o basept}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} L : \pi_1 X \to S_n \\ \text{w/ trans image} \\ \text{mod conj in } S_n \end{matrix}\right\}$$

<u>Recall</u>: Covering trans

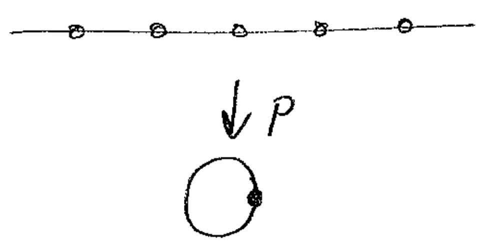
$$\begin{tikzcd} \tilde{X} \arrow[rr, "f"] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

f a homeo with $p \circ f = p$.

$G(\tilde{X})$ = group of such

(op. is comp. of fns)

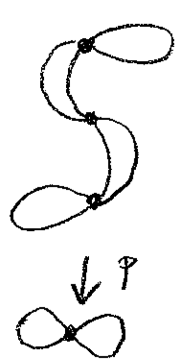
<u>Ex</u>:



$G(\tilde{X})$ = trans w/ int shifts

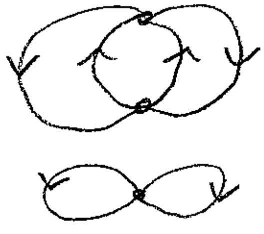
($\cong \pi_1 S^1$!)

<u>Ex</u>:



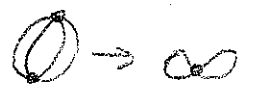
[Query:] $G(\tilde{X}) = \{ \mathrm{id}_{\tilde{X}} \}$

<u>Ex</u>:



$G(\tilde{X}) = \mathbb{Z}/2\mathbb{Z}$

<u>Def</u>: A connected cover is normal (or regular, or galois) if $\forall\, \tilde{x}_0, \tilde{x}_1 \in \tilde{X}$ with $p(\tilde{x}_0) = p(\tilde{x}_1)$, $\exists\, f \in G(\tilde{X})$ with $f(\tilde{x}_0) = \tilde{x}_1$.

[Ex: $\mathbb{R} \to S^1$, ]

<u>Note</u>: For X reasonable, such f exists iff $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = p_*(\pi_1(\tilde{X}, \tilde{x}_1))$

$$= \gamma\, p_*(\pi_1(\tilde{X}, \tilde{x}_1))\, \gamma^{-1}$$

<u>Thm</u>: X path conn, loc. path conn, S.L.S.C.

A connected cover $\tilde{X} \xrightarrow{p} X$ is regular $\iff$ $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ is a normal subgroup of $\pi_1(X, x_0)$.

<u>Next time</u>: $$\pi_1(X) \big/ p_*(\pi_1 \tilde{X}) \cong G(\tilde{X})$$