Lecture 24: Homology and homotopic maps.

Today:

Thm $f, g : X \to Y$ homotopic, then $f_* = g_* : H_n(X) \to H_n(Y)$.

Cor: If $X \simeq_{h.e.} Y$, then $H_n(X) \cong H_n(Y)$.

Cor: $H_n(\mathbb{R}^k) \cong H_n(pt) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$

Pf of Cor:

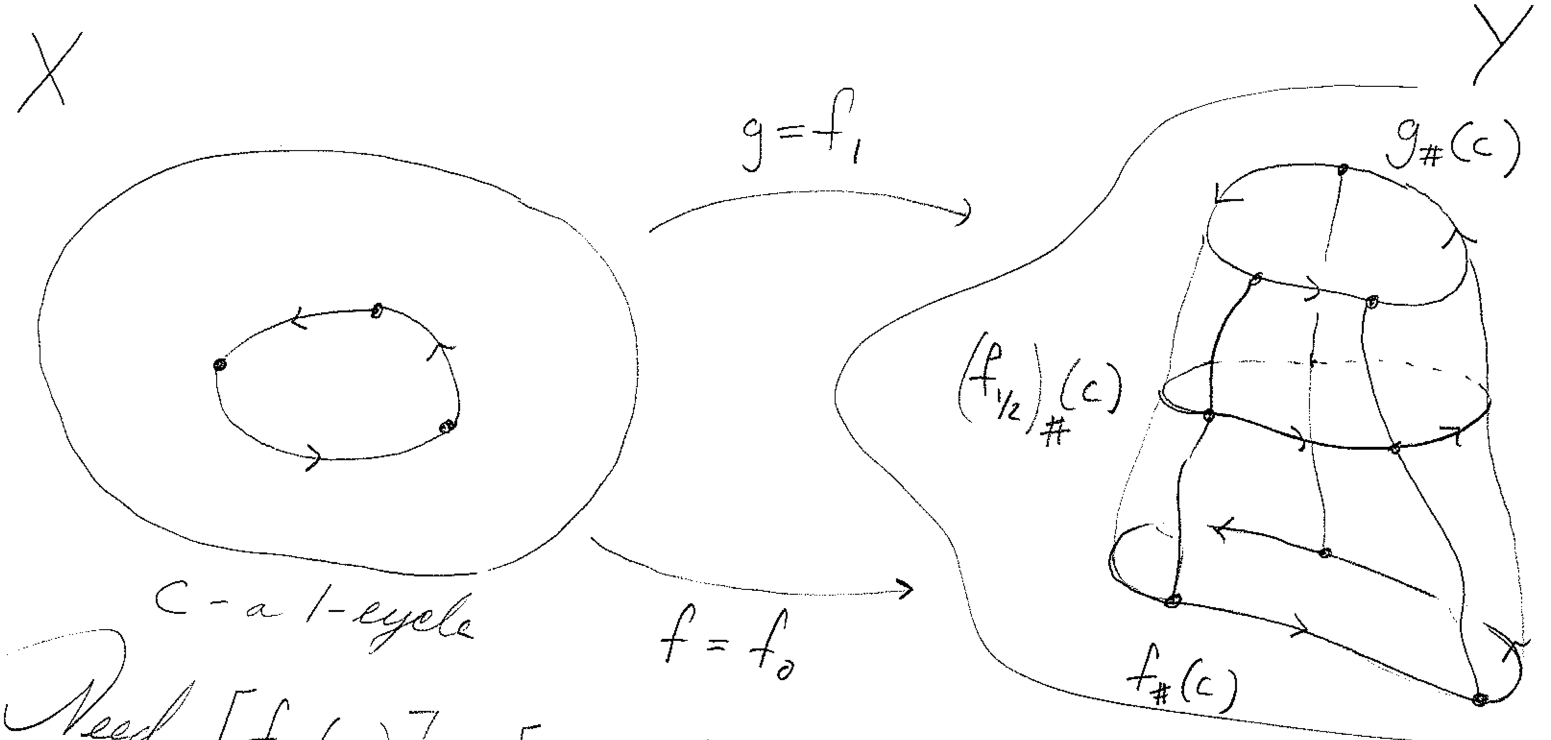
$$X \underset{g}{\overset{f}{\rightleftarrows}} Y \qquad \begin{array}{l} f \circ g \simeq id_Y \\ g \circ f \simeq id_X \end{array}$$

So

$$H_n(X) \underset{g_*}{\overset{f_*}{\rightleftarrows}} H_n(Y) \qquad \begin{array}{l} f_* \circ g_* \overset{\text{discuss}}{=} (f \circ g)_* = (id_Y)_* \\ \qquad = id_{H_n(Y)} \\ g_* \circ f_* = id_{H_n(X)} \end{array}$$

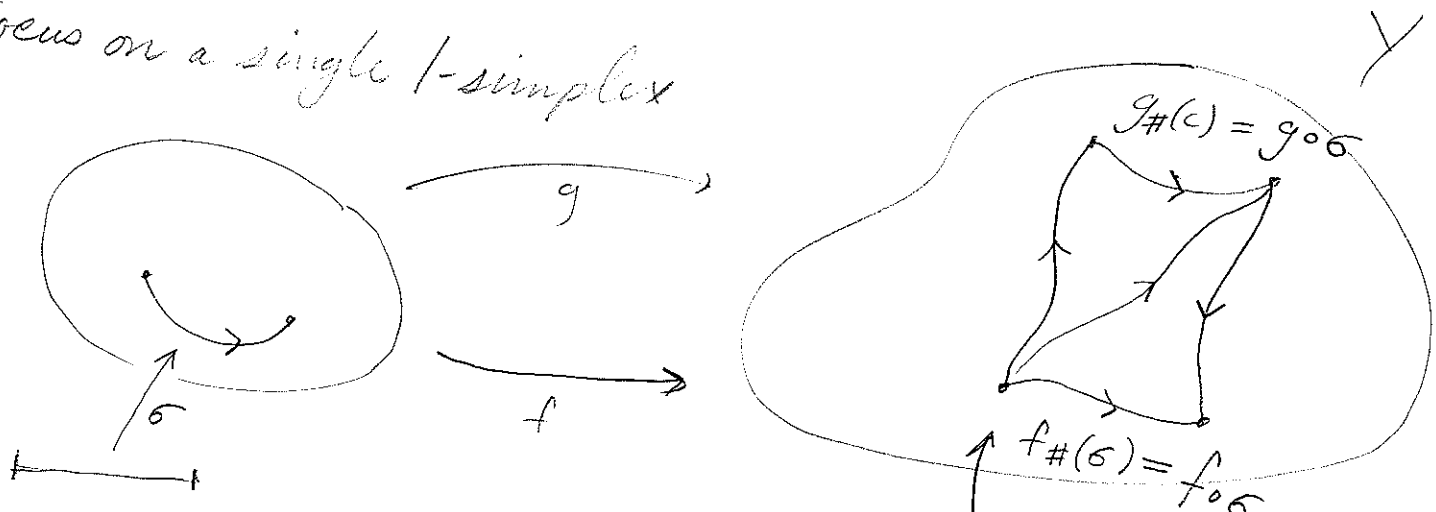
$\blacksquare$

Pf of Thm: Let $f_t : X \times I \to Y$ be a homotopy from $f = f_0$ to $g = f_1$.

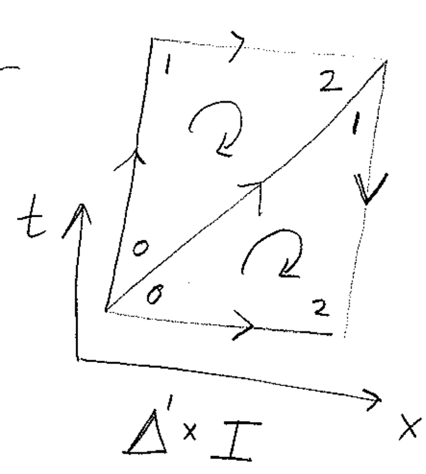


Need $[f_\#(c)] = [g_\#(c)]$ in $H_1(Y)$, i.e. $d \in C_2(Y)$ with $\partial d = g_\#(c) - f_\#(c)$.

Focus on a single 1-simplex



Have a homotopy from $f \circ \sigma$ to $g \circ \sigma$ via $(x,t) \longmapsto f_t(\sigma(x))$



Gives a 2-chain $P(\sigma)$

<u>Key:</u> $\partial P(\sigma) = g_\#(\sigma) - f_\#(\sigma) - P(\partial\sigma)$

Will show such exists for all σ

<u>Algebra Lemma:</u> Suppose

$$\begin{tikzcd} \arrow[r] & C_{n+1}(X) \arrow[r] \arrow[d, "f_\#"', "g_\#"] & C_n(X) \arrow[r, "\partial"] \arrow[dl, "P"'] \arrow[d, "f_\#"', "g_\#"] & C_{n-1}(X) \arrow[r] \arrow[dl, "P"'] \arrow[d, "f_\#"', "g_\#"] & {} \\ & C_{n+1}(Y) \arrow[r] & C_n(Y) \arrow[r] & C_{n-1}(Y) \arrow[r] & {} \end{tikzcd}$$

"Chain homotopy."

where $\partial \circ P + P \circ \partial = g_\# - f_\#$. Then $f_\#$ and $g_\#$ induce the same map on homology

<u>Pf.</u> Let $c \in C_n(X)$ be n-cycle. Then

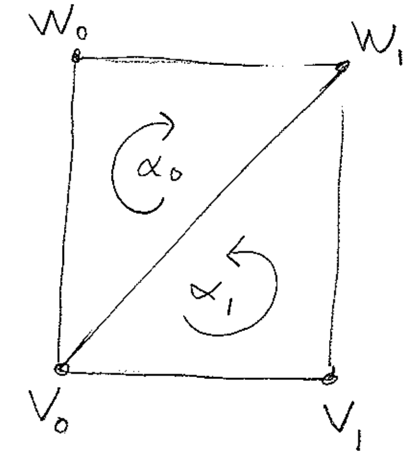
$$g_\#(c) - f_\#(c) = \partial P(c) + \underbrace{P(\overset{0}{\overset{\shortparallel}{\partial c}})}_{\text{Explain in the picture.}} = \partial P(c)$$

<u>Constructing P:</u> $\sigma : \Delta^n \to X$, consider $\Delta^n \times I \xrightarrow{F} Y$ the induced homotopy between $g_\#(\sigma)$ and $f_\#(\sigma)$

$[F = f_t \circ (\sigma \times id_I)]$

$[v_0, \dots, v_n] = \Delta^n \times \{0\}$

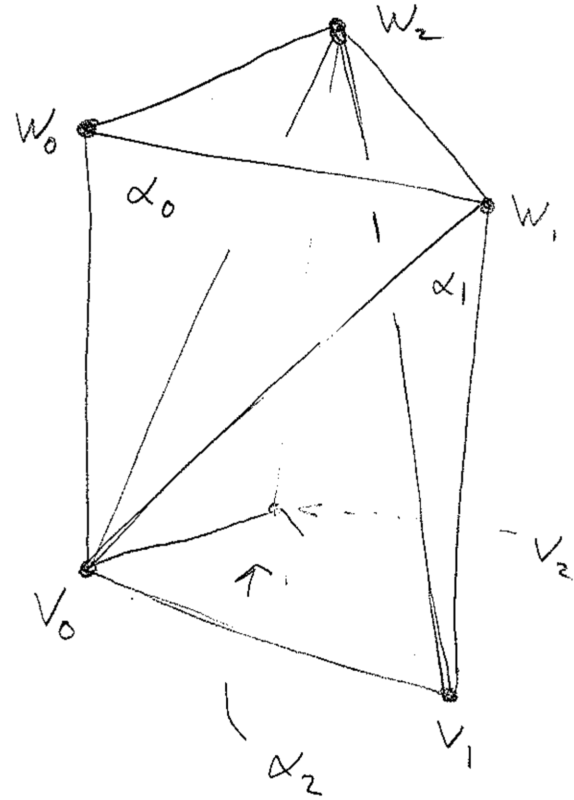
$[w_0, \dots, w_n] = \Delta^n \times \{1\}$



$\Delta^n \times I$ is the union of $(n+1)$-simplices

$\alpha_i = [v_0, \dots, v_i, w_i, w_{i+1}, \dots, w_n]$

for $i = 0, 1, \dots, n$.



Now define

$$P(\sigma) = \sum_{i=0}^{n} (-1)^i F\big|_{\alpha_i}$$

__Check:__ $P(\partial\sigma) + \partial P(\sigma) = g_\#(\sigma) - f_\#(\sigma)$

__Recall:__ $A^{\text{clsd}} \subseteq X$. If A is contractible (+ reasonable), then $X \to X/A$ is a homotopy equiv.

__Goal:__ $A^{\text{clsd}} \subseteq X$, want to relate $H_*(A)$, $H_*(X)$ and $H_*(X/A)$

[Know 2, figure out the third] __Ex:__ $X = D^n$, $A = \partial D^n = S^{n-1}$, $X/A = S^n$