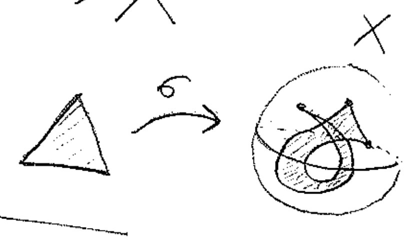
Lecture 23: Singular Homology

Last time on math 525:

Singular n-simplex: A map $\sigma : \Delta^n \to X$



Singular homology:

$C_n(X) =$ free abelian gp on all sing. n-simpl

$\partial_n : C_n(X) \to C_{n-1}(X)$ by $\partial_n \sigma = \sum_{k=0}^{n} (-1)^k \sigma\Big|_{[e_0, \ldots, \hat{e}_k, \ldots]}$

$H_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}$ [makes sense as $\partial_n \circ \partial_{n+1} = 0$]

$X = \{pt\}$

Cellular Homology:

$$C_2(X) \to C_1(X) \to C_0(X) \to 0$$

$$C_2(X) = 0, \quad C_1(X) = 0, \quad C_0(X) = \mathbb{Z}$$

$$H_n^{\Delta}(pt) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Singular Homology

$\exists!$ map $\sigma_n : \Delta^n \to X$, so $C_n(X) \cong \mathbb{Z}$.

$$\begin{array}{ccccccccc} C_3(X) & \xrightarrow{0} & C_2(X) & \xrightarrow{\cong} & C_1(X) & \xrightarrow{0} & C_0(X) & \longrightarrow & 0 \\ \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & & \end{array}$$

$$\partial_1 \sigma_1 = \sigma_2 - \sigma_2 = 0$$

$$\partial_2 \sigma_2 = \sigma_1 - \sigma_1 + \sigma_1 = \sigma_1$$

$$H_0(X) = \ker \partial_0 / \operatorname{im} \partial_1 \cong \mathbb{Z}$$

$$H_1(X) = \ker \partial_1 / \operatorname{im} \partial_2 = C_1(X)/C_1(X) = 0$$

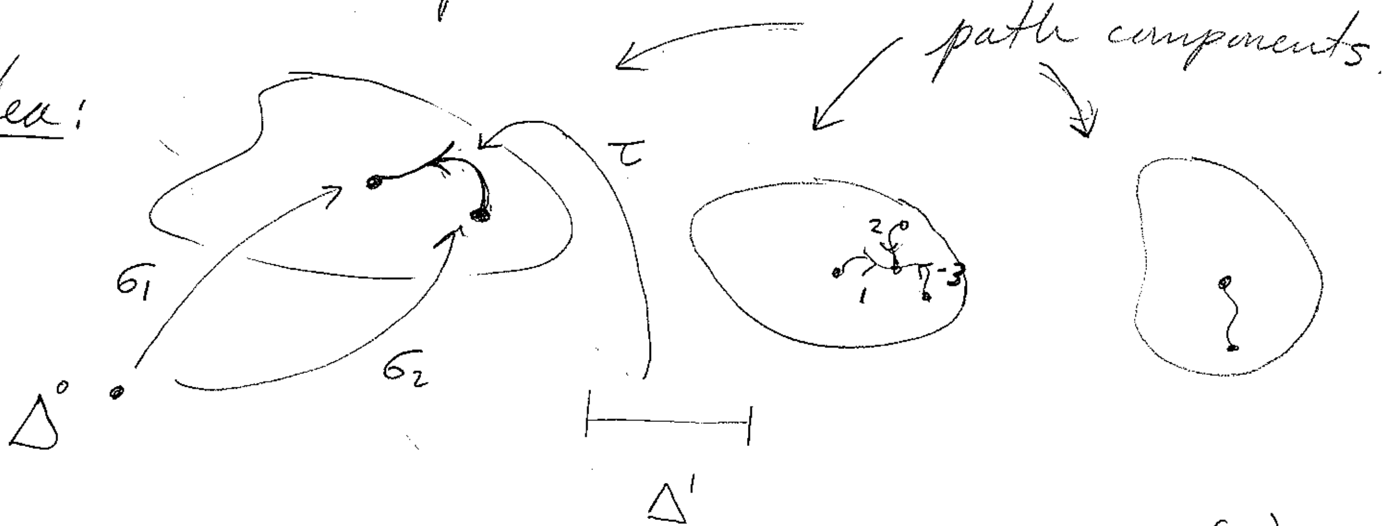
$$H_2(X) = \ker \partial_2 / \operatorname{im} \partial_3 = 0/0 = 0.$$

$$H_n(X) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

[Agrees with H^Δ, and this is true in general.]

Prop: $H_0(X) = \bigoplus_{\text{path comp.}} \mathbb{Z}$

Idea:



$$\partial \tau = \sigma_2 - \sigma_1$$

$$H_1(X) = C_0(X) / \operatorname{im} \partial_1$$

Thus $H_1(X) \cong \bigoplus_{\substack{\text{path} \\ \text{comp.}}} \mathbb{Z}$ ∎

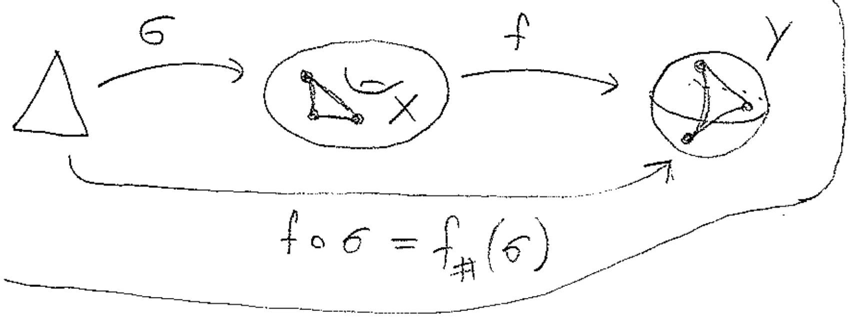
<u>Induced maps:</u> $f: X \to Y$ a map.

Define $f_*: H_n(X) \to H_n(Y)$ as follows.

draw first

$$\begin{tikzcd}
C_{n+1}(X) \arrow[r, "\partial_{n+1}"] \arrow[d, "f_\#"'] & C_n(X) \arrow[r, "\partial_n"] \arrow[d, "f_\#"] & C_{n-1}(X) \arrow[d, "f_\#"] & H_n(X) = \ker\partial_n / \operatorname{im}\partial_{n+1} \arrow[d, dashed, "f_*"] \\
C_{n+1}(Y) \arrow[r, "\partial'_{n+1}"] & C_n(Y) \arrow[r, "\partial'_n"] & C_{n-1}(Y) & H_n(Y) = \ker\partial'_n / \operatorname{im}\partial'_{n+1}
\end{tikzcd}$$

where $f_\#(\sigma: \Delta^n \to X) = (f \circ \sigma): \Delta^n \to Y$. For this to

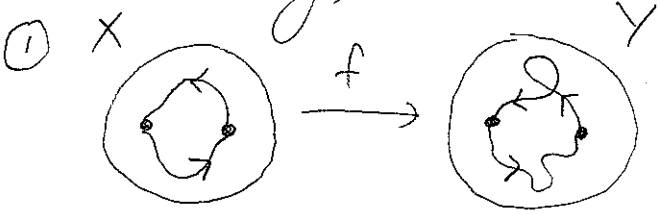


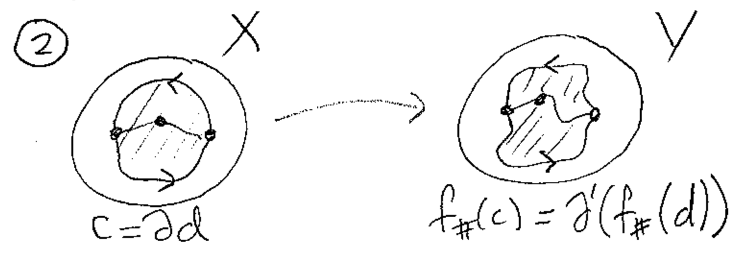
give a map on homology, need

① $f_\#(\ker\partial_n) \subseteq \ker\partial'_n$ $\left(\text{gives } \ker\partial_n \to H_n(Y)\right)$

② $f_\#(\operatorname{im}\partial_{n+1}) \subseteq \operatorname{im}\partial'_{n+1}$ $\left(\text{gives } H_n(X) \to H_n(Y)\right)$

Geometrically, this makes sense





__Claim:__ $f_\# \circ \partial_n = \partial_n' \circ f_\#$ (The diagram commutes.)

Alg., this implies ① and ② [Explain]

__Pf:__
$$
\begin{aligned}
f_\#(\partial_n \sigma) &= f_\#\Big(\sum_{k=0}^{n} (-1)^k \sigma\big|_{k^{th}\text{ face}}\Big)\\
&= \sum_{k=0}^{n} (-1)^k f_\#\big(\sigma\big|_{k^{th}\text{ face}}\big)\\
&= \sum_{k=0}^{n} (-1)^k f\circ\sigma\big|_{k^{th}\text{ face}} = \partial_n'(f\circ\sigma)\\
&= \partial_n'(f_\#(\sigma)) \qquad \blacksquare
\end{aligned}
$$

__Terminology:__

abelian gps

$$
\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f"] \arrow[dr, phantom, "\circlearrowleft"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f"] \arrow[dr, phantom, "\circlearrowleft"] & C_{n-1} \arrow[r] \arrow[d, "f"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\partial_n'"'] & D_n \arrow[r, "\partial_n'"'] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}
$$

} Chain complex if $\partial_n \circ \partial_{n+1} = 0$

A f make the diagram commute ($f \circ \partial_n = \partial_n' \circ f$) is called a __chain map__ $\Longrightarrow$ gives a map on homology.

Next time:

Thm: $f, g : X \to Y$ are homotopic. Then $f_* = g_*$.

Cor: $f : X \to Y$ is a homotopy equiv. Then
$f_* : H_n(X) \to H_n(Y)$ is an isomorphism.

Pf:

$$X \underset{g}{\overset{f}{\rightleftarrows}} Y \quad \text{where} \quad \begin{array}{l} g \circ f \simeq id_X \\ f \circ g \simeq id_Y \end{array}$$

So

Explain

$$H_n(X) \underset{g_*}{\overset{f_*}{\rightleftarrows}} H_n(Y) \qquad \begin{array}{l} g_* \circ f_* = (g \circ f)_* = (id_X)_* \\ \qquad = id_{H_n(X)} \\ f_* \circ g_* = id_{H_n(Y)} \end{array}$$

are inverse isomorphisms. $\blacksquare$

Cor: If X is contractible, e.g. $X = \mathbb{R}^n$, then
$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n > 0 \end{cases}$$