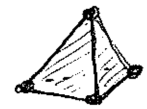
Lecture 22: Homology of Δ-complexes

Last time:

n-simplex: Convex hull of $n+1$ generic pts

$n=0$ $\cdot$ $n=1$ $\circ\!\!-\!\!\circ$ $n=2$  $n=3$ 

Standard n-simplex:

$$\Delta^n = \text{convex hull of } e_0, e_1, \ldots, e_n \text{ in } \mathbb{R}^{n+1}.$$

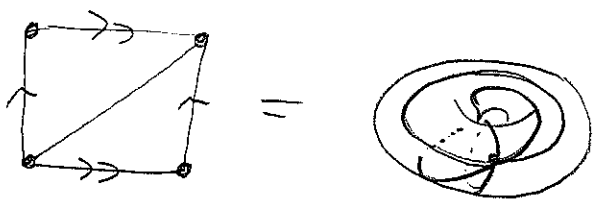
Orientation of a simplex: An ordering $[v_0, v_1, \ldots, v_n]$ of its vertices.

Δ-complex: Maps $\sigma_\alpha : \Delta^n \to X$ such that

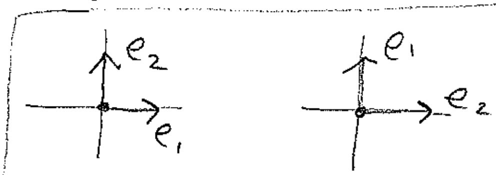
① $\sigma_\alpha|_{\mathring{\Delta}}$ is 1-1, $X = \coprod_\alpha \sigma_\alpha(\mathring{\Delta})$.

② F a face of Δ^n, then $\sigma_\alpha|_F$ some $\sigma_\beta : \Delta^{n+1} \to X$ where F is ident with Δ^{n-1} via the orient. pres. affine isom.

③ $U \subseteq X$ is open $\iff$ $\sigma_\alpha^{-1}(U)$ is open $\forall \alpha$.

Ex: 

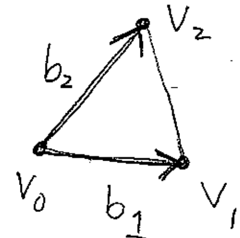
<u>Orient of $\mathbb{R}^n$</u>: choice of <u>ordered</u> basis $b_1, \ldots, b_n$;
same as $b_1', \ldots, b_n'$ if the change of basis matrix has pos. det.

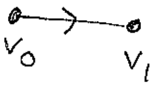


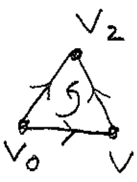
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

the two orients of $\mathbb{R}^2$

An orient $[v_0, v_1, \ldots, v_n]$ of Δ gives an orient of the cor. affine subspace via $\{v_1 - v_0, v_2 - v_0, \ldots, v_n - v_0\}$



$\partial($  $) = [v_1] - [v_0]$

$\partial($  $) = [v_1 v_2] - [v_0 v_2] + [v_0 v_1]$

A face $[w_0, \ldots, w_{n-1}]$ of Δ gets a $+1$
(outward normal to F, $w_1 - w_0, \ldots, w_{n-1}, w_0$) is a pos. orient basis for $\mathbb{R}^n$.

$$\partial([v_0, v_1, \ldots, v_n]) = \sum_{k=0}^{n} (-1)^k [v_0, \ldots, \underbrace{\hat{v}_k}_{\text{omit.}}, \ldots, v_n]$$

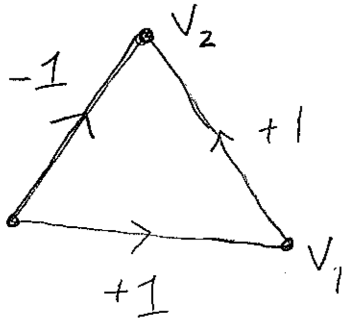
$C_n(X) =$ free ab. gp with basis $\sigma_\alpha : \Delta^n \to X$

$\partial_n : C_n(X) \to C_{n-1}(X)$ is defined by

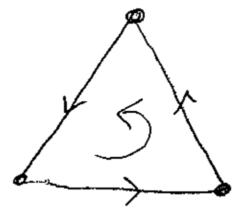
$$\partial_n(\sigma_\alpha) = \sum_{k=0}^{n} (-1)^n \sigma_\alpha\Big|_{[e_0,\dots,\hat{e}_k,\dots,e_n]}$$ ← ident with Δ^{n-1} in the order. pres. way.

<u>Lemma</u>: $\partial_{n-1} \circ \partial_n = 0$ [The boundary of something has no boundary.]

<u>Ex</u>:



where you should think



<u>Moral</u>: Lemma is really a fact about Δ^n, namely that $\partial\Delta^n = S^{n-1}$ has no boundary.

<u>Pf of Lemma</u>: Check that $\partial_{n-1}(\partial_n(\sigma_\alpha)) = 0$. ▨

<u>Cor</u>: $\operatorname{im}(\partial_{n+1}) \subseteq \ker \partial_n$

<u>Pf</u>: $C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X)$

<u>Upshot</u>: Can define $H_n^\Delta(X) = \ker \partial_n \big/ \operatorname{im} \partial_{n+1}$.

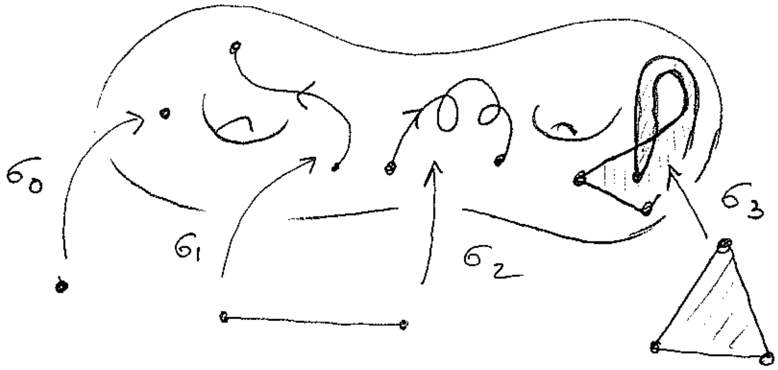
<u>Fact</u>: Only dep on X not the particular Δ-complex str. To see this we define homology for <u>any</u> space as follows.

X *any* top. space.

A *singular n-simplex* is a map $\sigma : \Delta^n \to X$

$C_n(X) =$ free ab. gp on *all* singular n-simplices

[very large!]



$\partial_n : C_n(X) \to C_{n-1}(X)$ by

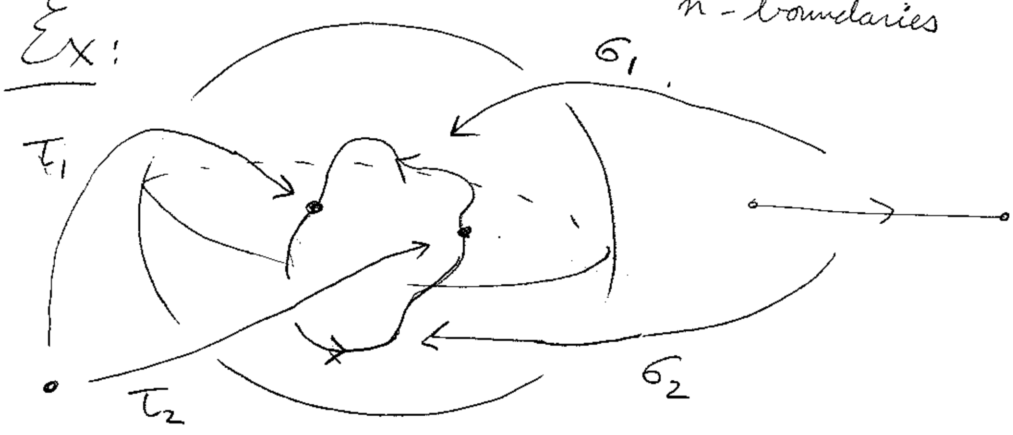
$$\partial_n(\sigma) = \sum_{k=0}^{n} (-1)^k \sigma\big|_{[e_0,\dots,\hat{e}_k,\dots,e_n]}$$

As before $\partial_{n-1} \circ \partial_n = 0$, so can define

$$H_n(X) = \underbrace{\ker \partial_n}_{n\text{-cycles}} \Big/ \underbrace{\operatorname{im} \partial_{n+1}}_{n\text{-boundaries}}$$

The *singular homology* of X.

Ex:

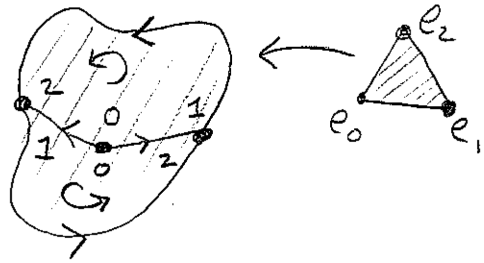


$\partial \sigma_1 = \tau_1 - \tau_2$
[not a 1-cycle]

$$\partial(\underbrace{\sigma_1 + \sigma_2}_{\text{a 1-cycle}}) = 0$$

It's also a 1-boundary:

[In fact $H_1(S^2) = 0$]



Ex: $X = \{pt\}$ [Q: How many singular n-simplices? A. One!]

$\exists!\ \sigma_n : \Delta^n \to X$

$$\xrightarrow{\cong} \underset{\phantom{\mathbb{Z}}}{C_3(X)} \xrightarrow{0} \underset{\mathbb{Z}}{\underset{\parallel}{C_2(X)}} \xrightarrow{\cong} \underset{\mathbb{Z}}{\underset{\parallel}{C_1(X)}} \xrightarrow{0} \underset{\mathbb{Z}}{\underset{\parallel}{C_0(X)}} \to 0$$

$$\left.\begin{array}{l} \partial\sigma_1 = \sigma_0 - \sigma_0 = 0 \\ \partial\sigma_2 = \sigma_1 - \sigma_1 + \sigma_1 = \sigma_1 \\ \quad\vdots \end{array}\right\} \Rightarrow \begin{array}{l} H_0(X) = \mathbb{Z} \\ H_n(X) = 0 \\ \text{for } n > 0. \end{array}$$

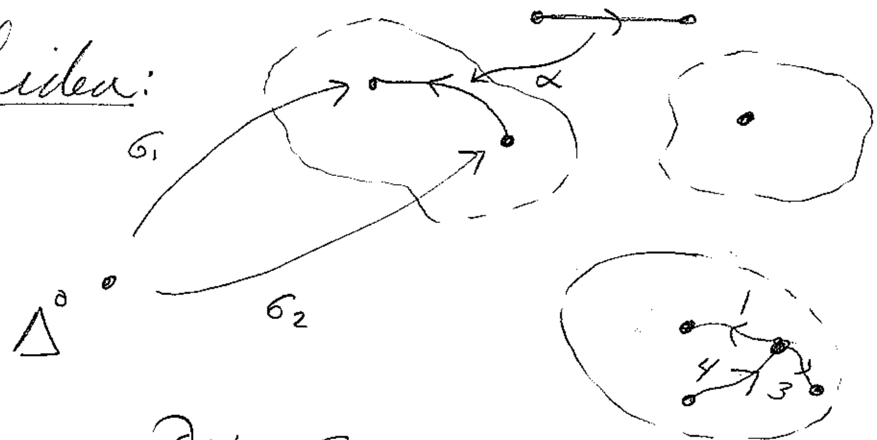
[Note: Same answer we'd get as a Δ-cplx.]

For any Δ cplx Y, will show $H_n^{\Delta}(Y) \cong H_n(Y)$

($\Rightarrow$ that H_n^{Δ} is indep of the Δ cplx str.)

Prop: $H_0(X) = \bigoplus_{\substack{\text{path} \\ \text{comp of } X}} \mathbb{Z}$

Pf idea:

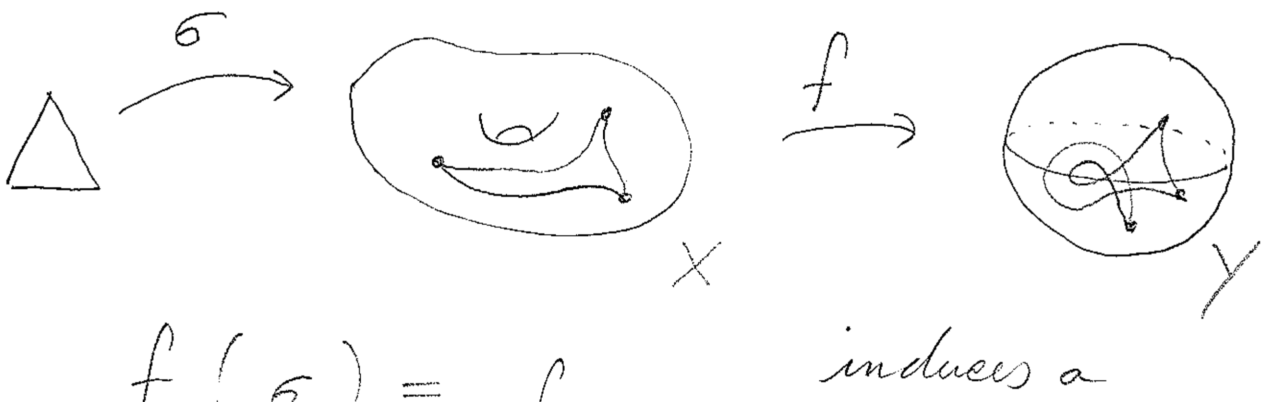


$H_0(X) = C_0(X) / \operatorname{im}\partial_1$

$\partial\alpha = \sigma_1 - \sigma_2$

$f: X \to Y$ a map. Next time, will define $f_*: H_n(X) \to H_n(Y)$.

Idea: $f_\#: C_n(X) \to C_n(Y)$



$f_\#(\sigma) = \underbrace{f \circ \sigma}_{\Delta^n \to Y}$ induces a map on $H_n(X)$.