Lecture 21:

Last time: X a CW complex

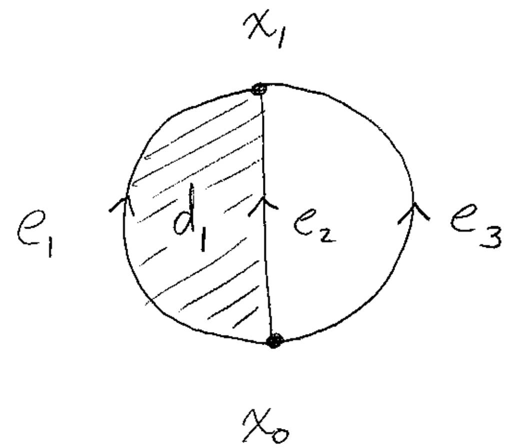
$C_n(X)$: free abelian gp with basis the n-cells of X.

$$\to C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} C_{n-2}(X)$$

homology

$$\boxed{H_n(X) = \ker \partial_n \big/ \operatorname{im} \partial_{n+1}} = \frac{\text{"n dim'l things w/o } \partial\text{"}}{\text{"boundaries of } n+1 \text{ dim'l things."}}$$

Ex:

$X =$ 

$C_0(X) = \{ a_0 x_0 + a_1 x_1 \mid a_i \in \mathbb{Z} \}$
$\cong \mathbb{Z}^2$

$C_1(X) = \{ c_1 e_1 + c_2 e_2 + c_3 e_3 \}$
$\cong \mathbb{Z}^3$

$C_2(X) = \{ b_1 d_1 \} \cong \mathbb{Z}$

[Start lecture here.]

Boundary maps:

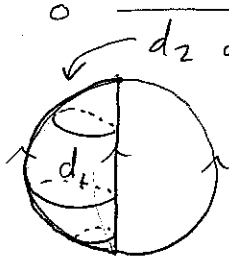
$\partial e_i = X_1 - X_0$

$\partial d_1 = e_2 - e_1$

$$H_0(X) = C_0(X) / \operatorname{im}\partial_1 \cong \mathbb{Z} = \mathbb{Z}^{\#(\text{path comp})}$$

$$H_1(X) = \ker\partial_1 / \operatorname{im}\partial_2 = \{c_1 + c_2 + c_3 = 0\} / e_2 - e_1 \cong \mathbb{Z} = (\pi_1 X)^{ab}$$

$$H_2(X) = \ker\partial_2 = 0$$

Consider instead $X =$ 

$$C_2(X) = \{ b_1 d_1 + b_2 d_2 \} \cong \mathbb{Z}^2$$

$$H_2(X) = \ker\partial_2 = \langle d_1 - d_2 \rangle \cong \mathbb{Z}$$

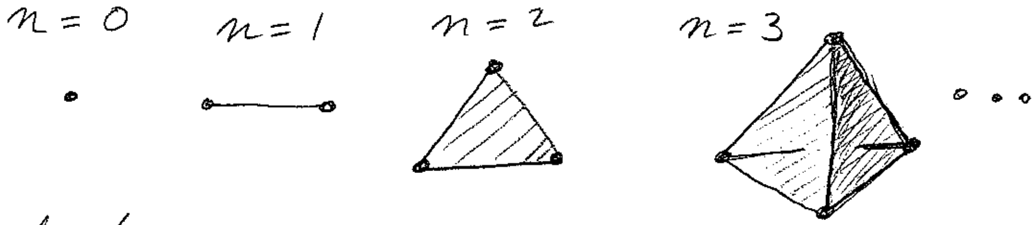
finally, some higher dim'l info!

Key Point: $H_n(X)$ depends only on (the homotopy type) of X, not the particular cell decomp.

Hard part: defining ∂_n

Solution: Use a restricted class of CW complexes, called Δ-complexes.

<u>n-simplex:</u>



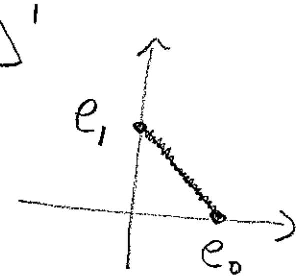
$\cdots$ [Convex hull of $n+1$ generic pts in $\mathbb{R}^m$]

<u>Standard n-simplex:</u>

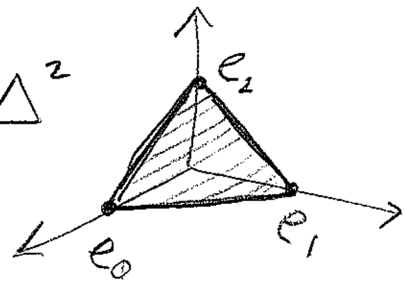
$$\Delta^n = \text{Convex hull of } e_0, e_1, \ldots, e_n \text{ in } \mathbb{R}^{n+1} = \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \;\middle|\; \begin{array}{l} \sum t_i = 1 \\ t_i \geq 0 \end{array} \right\}$$

A <u>face</u> of a simplex Δ is a subsimplex spanned by all but one of its vertices.

Δ^1



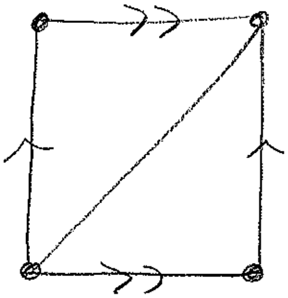
Δ^2

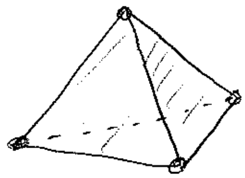


<u>Boundary</u>: $\partial\Delta$ = union of all the faces. | <u>Interior</u>:

$$\mathring{\Delta} = \Delta \setminus \partial\Delta$$

[Now consider the restricted class of CW complexes where the cells are simplices and the attaching maps glue each face of Δ^n to an $n-1$ simplex via a "nice" map.]

__Ex:__ $T =$  $=$ 

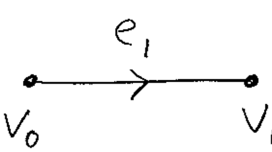
$S^2 =$  $= \partial\Delta^3$

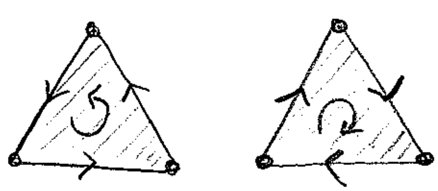
__Def:__ A Δ-complex str on X is a collection of maps $\sigma_\alpha : \Delta^n \longrightarrow X$ [n dep on α] such that:

(1) $\sigma_\alpha|_{\mathring{\Delta}^n}$ is injective. Each $x \in X$ is in the image of __exactly__ one σ_α.

(2) If F is a face of Δ^n, then $\sigma_\alpha|_F$ is one of the $\sigma_\beta : \Delta^{n-1} \longrightarrow X$, after ident. F with Δ^{n-1} via a linear map

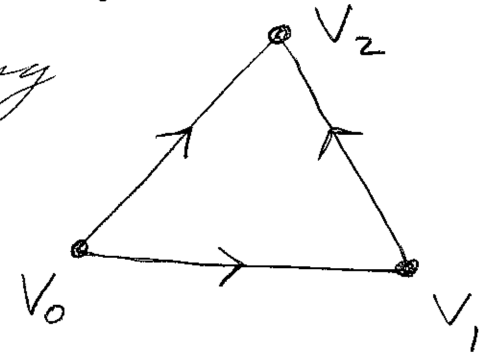
(3) $A \subseteq X$ is open $\Longleftrightarrow$ $\sigma_\alpha^{-1}(A)$ is open for all α.

__Orientations:__ 



$\partial e_1 = v_1 - v_0$

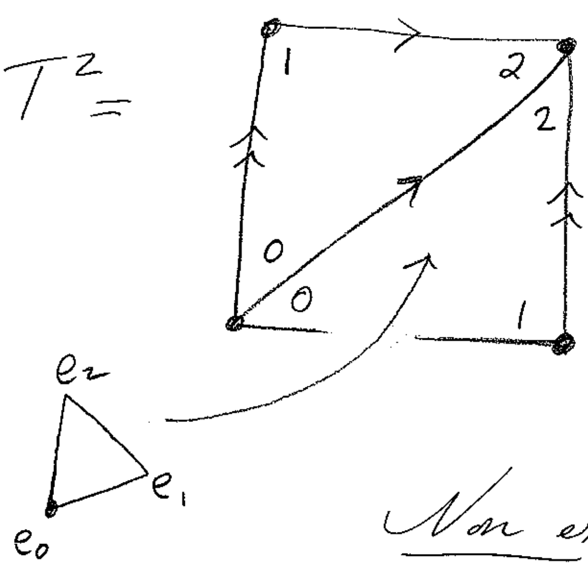
An <u>oriented simplex</u> is a simplex together w/ an ordering of its vertices $[v_0, v_1, \ldots, v_n]$.
[Fixes an ident with Δ^n.] A face gets an orient just by restricting the ordering



<u>Addendum to 2:</u>

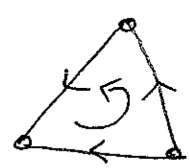
The ident of F with Δ^{n-1} pres the ordering of the vertices.

$T^2 =$



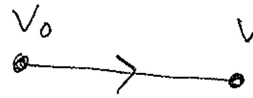
[Not so important in this class, but is crucial for defining the product on cohomology in 526]

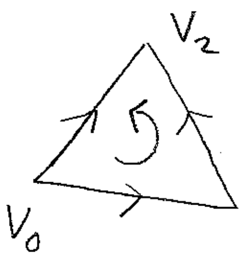
<u>Non ex:</u>

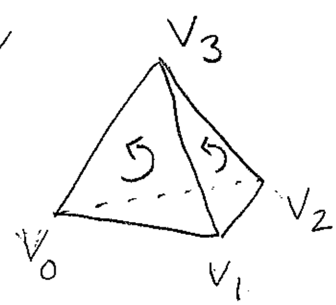


Homology of a Δ-cplx:

$C_n(X)$ = free ab. gp on $\sigma_\alpha : \Delta^n \to X$.

$\partial\Big($  $\Big) = [v_1] - [v_0]$

$\partial\Big($  $\Big) = [v_1 v_2] - [v_0 v_2] + [v_0 v_1]$

$\partial\Big($  $\Big) = [v_1\, v_2\, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2]$

[Explain via right hand rule]

Def: $$\partial_n(\sigma_\alpha) = \sum_{i=0}^{n} (-1)^i \sigma_\alpha \Big|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$$

means omit

If time remains, discuss orient of $\mathbb{R}^n$ in terms of equiv. clss of basis.