Lecture 19:

<u>Last time:</u> $(\tilde{X}, \tilde{x}_0) \to (X, x_0)$ a path conn. regular cover

$$\tau : \pi_1(X, x_0) \longrightarrow G(\tilde{X}) = \text{group of covering transformations}$$

$$\alpha \longmapsto$$ covering trans taking x_0 to $\tilde{\alpha}(1)$ where $\tilde{\alpha}$ is the lift of α starting at $\tilde{x}_0$.

<u>Thm</u>: τ is onto and $\ker \tau = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$, i.e.

$$\pi_1 X / \pi_1 \tilde{X} = G(\tilde{X}).$$

<u>Note</u>: τ gives a map

$$\pi_1(X, x_0) \longrightarrow \mathrm{Sym}\left(p^{-1}(x_0)\right)$$

$$\alpha \longmapsto$$ action of τ_α on $p^{-1}(x_0)$.

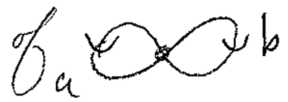
Also have the lifting permutation

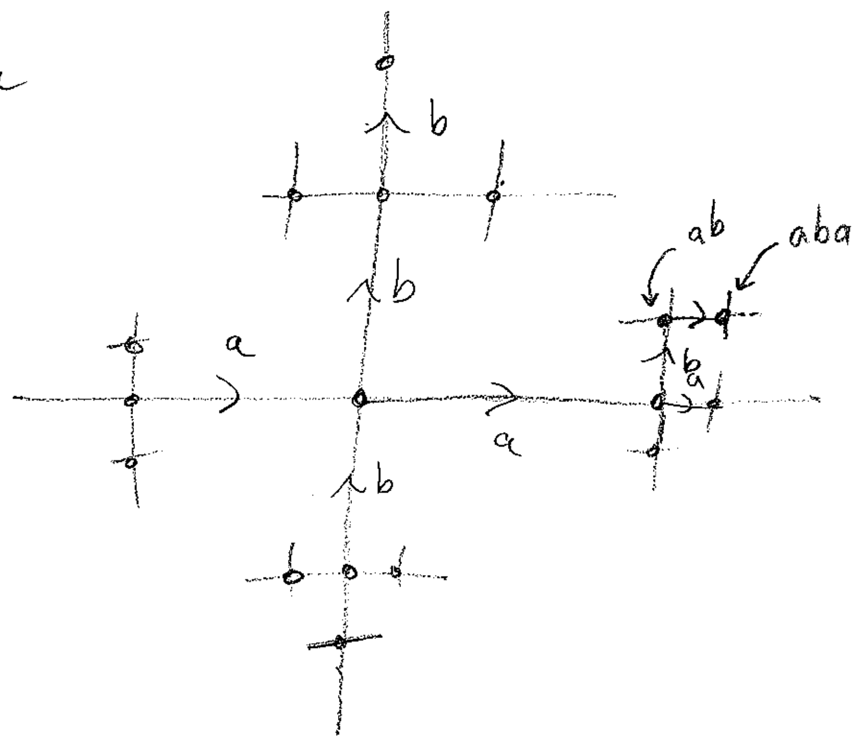
$$\pi_1(X, x_0) \longrightarrow \mathrm{Sym}(p^{-1}(x_0))$$
$$\alpha \longmapsto L_\alpha$$

where $L_\alpha(\tilde{x}) = \tilde{\alpha}(0)$ where $\tilde{\alpha}$ is the lift of α <u>ending</u> at $\tilde{x}$.

<u>Important note:</u> These are <u>not</u> related.

<u>Ex:</u> $\tilde{X}$ = Univ. Cover of





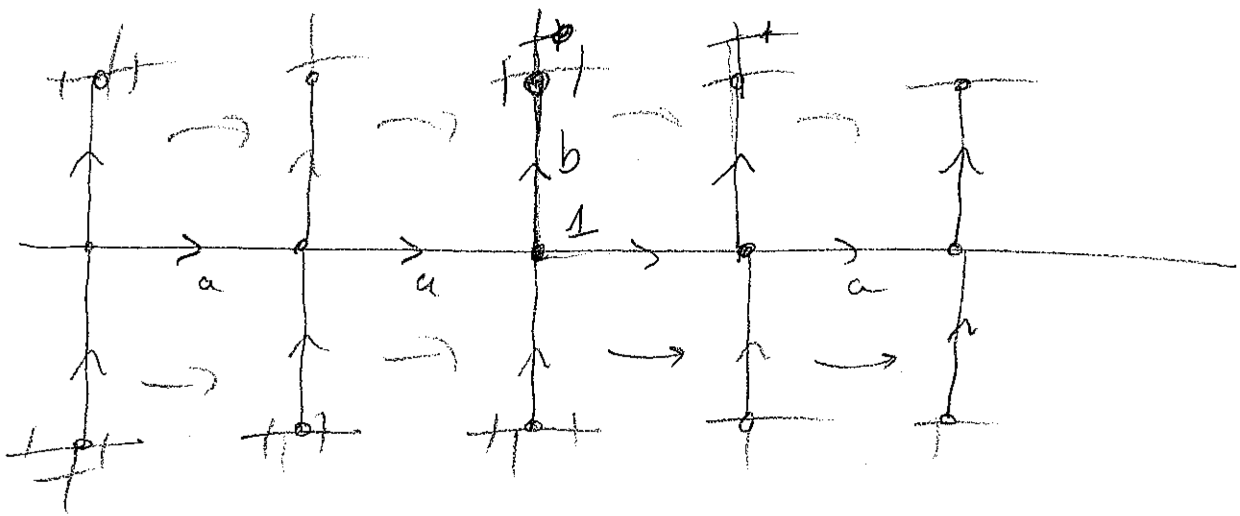
<u>Vertices:</u> elts of Free Gp(a,b)

<u>Edge:</u> between
w and wa
w and wb

Example of a <u>Cayley Graph</u>.

<u>Action of L_a:</u> Moves any vertex one unit to the left. In particular, any pt moves at most one unit.

<u>Action of τ_a:</u>



is a "translation" along this axis.

In particular $d(w, \tau_a(w))$ can be arb. large.

Covers as quotients:

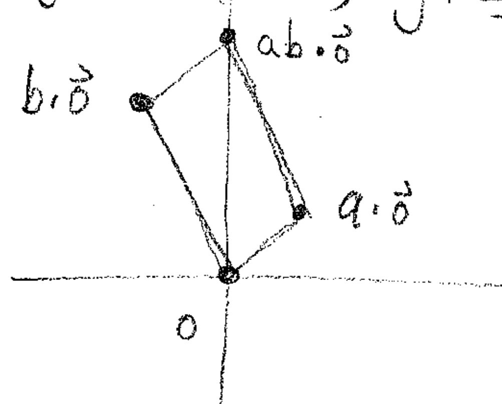
Thm: $\tilde{X}$ the univ. cover of X. If $H \leq \pi_1 X$, then $\tilde{X}/H$ is the cover corr to H.

Group action:

An action of a group G on a space X is a map
$$\begin{array}{ccc} G \times X & \longrightarrow & X \\ g \quad x & & g \cdot x \end{array}$$
sat. the usual rules, and <u>cont in the second input.</u>

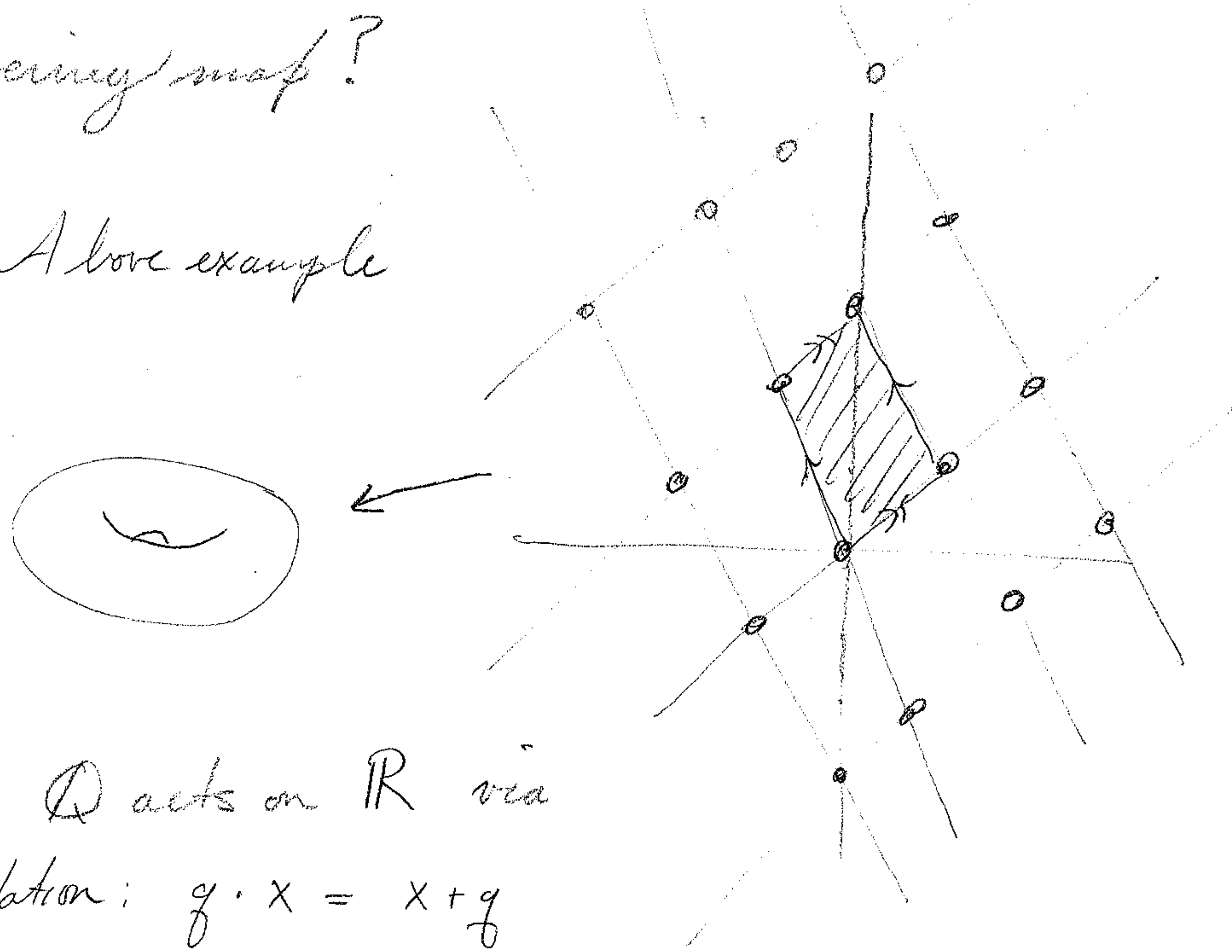
Ex: $G = \mathbb{Z}^2 = \langle a, b \rangle$ acts on $\mathbb{R}^2$ via

$$a \cdot (x, y) = (x+1, y+1)$$
$$b \cdot (x, y) = (x-1, y+2)$$



Suppose G acts on X. Give X/G the quotient top. When is $X \to X/G$ a covering map?

<u>Yes</u>: Above example



<u>No</u>: $\mathbb{Q}$ acts on $\mathbb{R}$ via translation: $q \cdot x = x + q$

Consider

$\mathbb{R}/\mathbb{Q}$ it has the <u>indiscrete</u> <u>top</u>: the only open sets are $\emptyset$ and the whole sp.

<u>Pf</u>: By def, if $U \subseteq \mathbb{R}/\mathbb{Q}$ is open then

$p^{-1}(U)$ is open in $\mathbb{R}$. So it contains an open interval, but its also $\mathbb{Q}$-invariant $\Rightarrow p^{-1}(U) = \mathbb{R} \Rightarrow U = \mathbb{R}/\mathbb{Q}$.

<u>Fact</u>: $X \to X/G$ is a covering map iff

① <u>G acts freely</u>: i.e. $g \cdot x = x \Rightarrow g = id$.

② <u>Orbits don't accumulate</u>: $\forall x \in X$, $\exists$ a nbhd U of x s.t. all $g \cdot U$ for $g \in G$ are disjoint. That is $g_1 U \cap g_2 U \neq \emptyset \Rightarrow g_1 = g_2$.

<u>Ex</u>: $X = \mathbb{H}^2 = \{ z \in \mathbb{C} \mid \mathrm{Im}(z) > 0 \}$

$$G = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle$$

$X/G =$