# Lecture 29: Some Applications 

Recall:

$$\tilde{H}_k(S^n) = \begin{cases} \mathbb{Z} & n = k \\ 0 & \text{otherwise} \end{cases} \Rightarrow S^n \not\simeq S^m \text{ if } n \neq m.$$

Thm: $\mathbb{R}^n \not\cong \mathbb{R}^m$ if $n \neq m$

Pf: $\mathbb{R}^n \setminus pt \simeq S^{n-1}$!

Def: A Hausdorff top. space $X$ with a countable basis is an n-manifold if every $x \in X$ has a open nbhd $U \cong \mathbb{R}^n$.

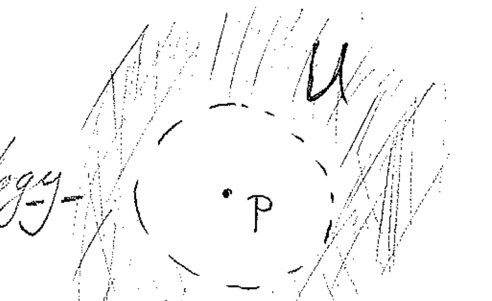
Ex: $\mathbb{R}^n$, $U \underset{\text{open}}{\subseteq} \mathbb{R}^n$, $S^n$, $T^n = S^1 \times \cdots \times S^1$.

[$n=2$ case is called a surface which you encounted on the HW. I work on $n=3$.]

Invariance of Dimension: $M$ an $m$-mfld.
$N$ an $n$-mfld.

If $M \cong N$, then $m = n$.

Proof: Pick $p$ in $M$, with a nbhd $U \cong \mathbb{R}^m$. Consider the local homology




$$H_k(M, M \setminus p) \underset{\substack{\text{by excision} \\ \text{with } Z = M \setminus U.}}{\cong} H_k(U, U \setminus p) \cong H_k(\mathbb{R}^m, \mathbb{R}^m - \{\vec{0}\})$$

Now $(\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}) \simeq_{h.e.} (D^m, D^m \setminus \{0\}) \underset{h.e.}{\simeq} (D^m, \partial D^m)$,

and so $H_k(M, M \setminus p) \cong H_k(D^m, \partial D^m) \cong \tilde{H}_k(S^m) = \begin{cases} \mathbb{Z} & m=k \\ 0 & \text{otherwise} \end{cases}$

Can do the same for $n$, and so $n = m$. ▨

Invariance of Domain: $U \overset{\text{open}}{\subseteq} \mathbb{R}^n$, $V \subseteq \mathbb{R}^n$.
If $U \cong V$, then $V$ is open.

Pf: Hatcher §2.B.3.

Brouwer Fixed Pt Thm: Let $f: D^n \to D^n$ be cont.
Then $\exists\, x \in D^n$ with $f(x) = x$.

Pf: If $f$ has no fixed pts, then can construct a retract $r: D^n \to \partial D^n$. Then $r_* \circ i_* = id_*$

$$\begin{array}{ccccc} H_{n-1}(\partial D^n) & \xrightarrow[i_*]{} & H_{n-1}(D^n) & \xrightarrow{r_*} & H_{n-1}(\partial D^n) \\ \mathbb{Z} & & 0 & & \mathbb{Z} \end{array}$$

a contradiction. ▨

<u>Degree</u>: $f: S^n \to S^n$ some map.

$$\mathbb{Z} \cong H_n(S^n) \xrightarrow{f_*} H_n(S^n) \cong \mathbb{Z}$$

$\alpha$ a gen $\qquad$ $(\deg f)\,\alpha$

[Doesn't depend on choice of $\alpha$.]

<u>Basic Facts</u>:

① $\deg(id_{S^n}) = 1$

② $f$ not onto $\Rightarrow \deg f = 0$ since if $p \notin f(S^n)$ then

$$\begin{tikzcd} H_n(S^n) \arrow[r, "f_*"] \arrow[rr, bend right, "f_*"'] & H_n(S^n \setminus p) \arrow[r, "i_*"] & H_n(S^n) \end{tikzcd}$$

$H_n(S^n \setminus p) \cong 0$

③ $f \simeq g \Rightarrow \deg f = \deg g$.

④ Every $k \in \mathbb{Z}$ is the degree of a map $S^n \to S^n$ for $n \geq 1$.

For $n = 1$, $\quad z \mapsto z^k$ has deg $= k$

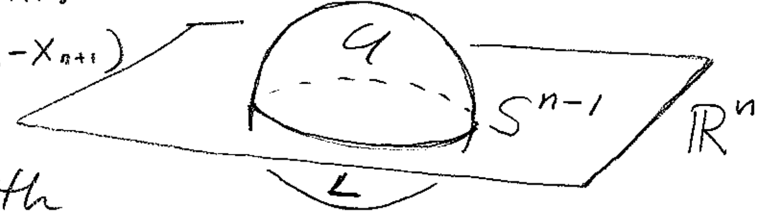
$\qquad\qquad\quad z \mapsto \bar{z}^k$ has deg $= -k$

For $n > 1$ will give examples later.

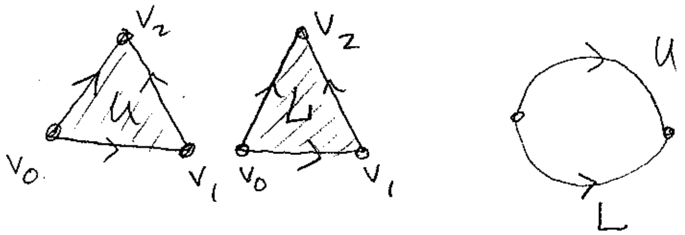
<u>Note</u>: Turns out that the converse of ③ is true, i.e. $\deg f = \deg g \Rightarrow f \simeq g$, i.e. $\pi_n S^n \cong \mathbb{Z}$.

⑤ Suppose $f$ is reflection in a plane through 0.
Then $\deg f = -1$.   $f(x_1, \dots, x_{n+1}) = (x_1, \dots, x_n, -x_{n+1})$

$S^n \subseteq \mathbb{R}^{n+1}$




$S^n$ has a $\Delta$-complex str. with two $n$-simplices



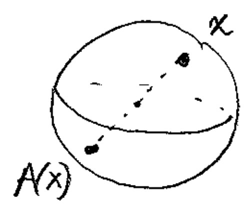

$H_n(S^n)$ is gen by $[U - L]$ and

$$U - L \xrightarrow[f_\#]{} L - U = -(U - L) \Rightarrow \deg f = -1.$$

⑥ $\deg(f \circ g) = \deg(f)\deg(g)$

$$\begin{array}{ccccc} H_n(S^n) & \xrightarrow{g_*} & H_n(S^n) & \xrightarrow{f_*} & H_n(S^n) \\ \underset{\text{a gen}}{\alpha} & \longmapsto & (\deg g)\alpha & \longmapsto & (\deg f)(\deg g)\alpha \end{array}$$

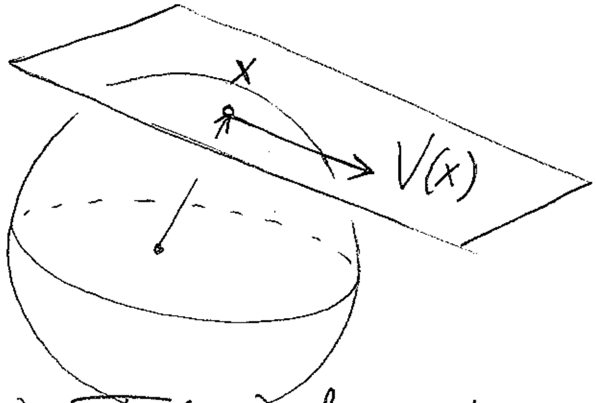
⑦ Antipodal map: $A : S^n \to S^n$   $A(x) = -x$




$A = \left(\begin{matrix}\text{comp of} \\ n+1 \text{ reflections}\end{matrix}\right) \Rightarrow \deg A = (-1)^{n+1}$

$\mathbb{RP}^n \cong S^n / x \sim A(x)$

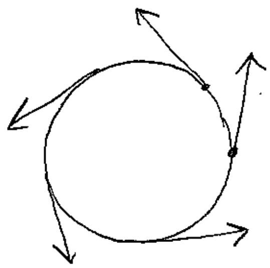
## Vector Fields on $S^n$:



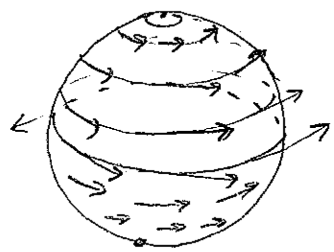

$$T_x S^n = \{ v \in \mathbb{R}^{n+1} \mid x \cdot v = 0 \}$$

$V : S^n \to \mathbb{R}^{n+1}$ s.t. $V(x) \in T_x(S^n)$ for each $x$.

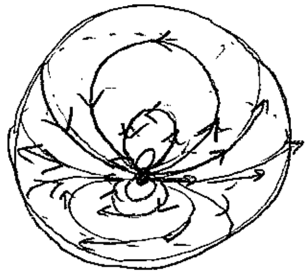
Ex:



$n = 1$

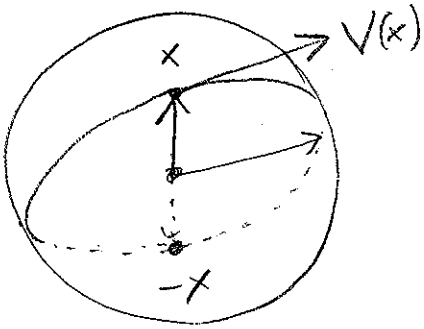


$n=2$: Vanishes at poles.



**Thm:** $S^n$ has a nowhere vanishing vector field $\Leftrightarrow$ $n$ is odd.

*Proof:* ($\Rightarrow$) Suppose $V$ is such a vector field. Rescale $V$ so that it has unit length, i.e. $V' = \dfrac{V(x)}{|V(x)|}$.




Consider $f_t : S^n \times [0,1] \to S^n$

$$f_t(x) = \cos(\pi t)\, x + \sin(\pi t)\, V(x)$$

$$f_0 = id_{S^n} \quad \text{and} \quad f_1 = A$$

So $id \simeq A \Rightarrow 1 = \deg id = \deg A = (-1)^{n+1} \Rightarrow n$ odd.

($\Leftarrow$) $V(x_1, x_2, \ldots, x_{n+1}) = (-x_2, x_1, -x_4, x_3, \ldots, -x_{n+1}, x_n)$ ∎