Lecture 28: Equality of homologies

X a Δ-complex

Simplicial Homology: $H_n^\Delta(X)$ – easy to compute.
– seems to depend on cellulation.

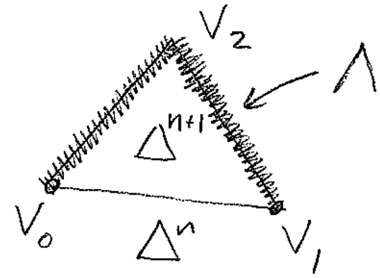
Singular Homology: $H_n(X)$ – clearly invariant.

Thm: $H_n^\Delta(X) \cong H_n(X)$

Lemma: $H_n(\Delta^n, \partial\Delta^n) \cong \mathbb{Z}$ is generated by $i_n = id_{\Delta^n}$.

Pf: Induct on n. Clear for $n=0$, i.e. $(pt, \emptyset)$.

Λ = All faces of Δ^{n+1} except the 1st.



By the long exact seq of $(\Lambda, \partial\Delta^{n+1}, \Delta^{n+1})$:

$$\begin{tikzcd}
\to H_{n+1}(\Delta^{n+1}, \Lambda) \arrow[r] \arrow[d, equal] & H_{n+1}(\Delta^{n+1}, \partial\Delta^{n+1}) \arrow[r, "\partial", "\cong"'] & H_n(\partial\Delta^{n+1}, \Lambda) \arrow[r] & H_n(\Delta^{n+1}, \Lambda) \arrow[d, equal] \\
0 & & H_n(\Delta^n, \partial\Delta^n) \cong \mathbb{Z} \arrow[u, "i_*", "\cong"'] & 0
\end{tikzcd}$$

as $\Delta^{n+1}/\Lambda \simeq pt.$

$H_n(\Delta^n, \partial\Delta^n) \cong \mathbb{Z}$ gen by i_n

<u>Claim</u>: $\partial[i_{n+1}] = i_*[i_n]$

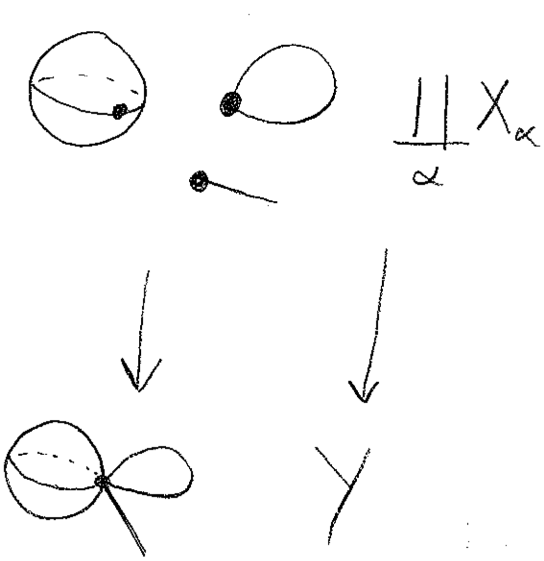
On the chain level,

$$\underset{C_{n+1}(\Delta^{n+1},\,\partial\Delta^{n+1})}{i_{n+1}} \xmapsto{\quad\partial\quad} \underset{C_n(\partial\Delta^{n+1},\,\Lambda)}{\sum_{k=0}^{n+1}(-1)^k\, i_{n+1}\Big|_{k^{th}\text{ face}} = i_n}$$

So $H_{n+1}(\Delta^{n+1},\partial\Delta^{n+1})$ is gen by i_{n+1} ∎

<u>Lemma</u>: $x_\alpha \in X_\alpha$ s.t. (X_α, x_α) is a good pair. If $Y = \bigvee_\alpha X_\alpha$ and $i_\alpha : X_\alpha \to Y$ are the inclusions, then $\bigoplus_\alpha i_\alpha : \bigoplus_\alpha \tilde{H}_n(X_\alpha) \longrightarrow \tilde{H}_n(Y)$ is an isomorphism.

Pf:



By excision

$$\begin{array}{ccc} H_n\left(\coprod_\alpha X_\alpha, \coprod \{x_\alpha\}\right) & \underset{\cong}{\xrightarrow{q_*}} & \tilde{H}_n(Y) \\ \| & & \\ \bigoplus_\alpha H_n(X_\alpha, \{x_\alpha\}) & & \\ \| & \longleftarrow \text{induced by inclusion} & \\ \bigoplus_\alpha \tilde{H}_n(X_\alpha) & & \end{array}$$

∎

X w/ Δ-cplx str $\{ \sigma_\alpha : \Delta^{n_\alpha} \to X$ cell maps $\}$

Then we have a chain map $\quad C_n^\Delta(X) \longrightarrow C_n(X)$

gen by σ_α $\qquad$ gen by all $\Delta^n \to X$.

<u>Thm</u>: $H_n^\Delta(X) \longrightarrow H_n(X)$ is an isomorphism.

<u>Pf</u>: Assume X is finite dim'l [Full case in Hatcher.]

Inductively, show $H_*^\Delta(X^k) \cong H_*(X^k)$ $\quad$ k-skeleton, the union of all cells through dim k.

Base case $X^0 = \{\text{pts}\}$ is clear.

Assume true for k. Have

$$\begin{tikzcd}
H_{n+1}^\Delta(X^{k+1},X^k) \arrow[r] \arrow[d, "\cong"'] & H_n^\Delta(X^k) \arrow[r] \arrow[d, "\cong"'] & H_n^\Delta(X^{k+1}) \arrow[r] \arrow[d, "?"] & H_n^\Delta(X^{k+1},X^k) \arrow[r] \arrow[d, "\cong"'] & H_{n-1}^\Delta(X^k) \arrow[d, "\cong"] \\
H_{n+1}(X^{k+1},X^k) \arrow[r] & H_n(X_k) \arrow[r] & H_n(X^{k+1}) \arrow[r] & H_n(X^{k+1},X^k) \arrow[r] & H_{n-1}(X^k)
\end{tikzcd}$$

Now $\quad X^{k+1}/X^k = \bigvee_\alpha S^{k+1}$ $\quad$ with one sphere for each $k+1$ cell.

and $\quad \tilde{H}_n(S^{k+1}) = \begin{cases} \mathbb{Z} & n = k+1 \\ 0 & \text{otherwise} \end{cases}$

$$\text{So } H_n(X^{k+1}, X^k) = \begin{cases} \overbrace{\bigoplus_{\alpha} \mathbb{Z}}^{\text{sum over the } k+1 \text{ cells}} & n = k+1 \\ 0 & \text{otherwise} \end{cases}$$

Same is true for $H_n^{\Delta}(X^{k+1}, X^k)$ as the only non-zero chain group is $C_n^{\Delta}(X^{k+1}, X^k)$
$= \bigoplus_{\alpha} (\mathbb{Z}, \text{ gen by } \sigma_\alpha)$

Moreover

$$\begin{array}{c} H_n^{\Delta}(X^{k+1}, X^k) = \bigoplus_{\alpha} (\mathbb{Z}, \text{ gen by } \sigma_\alpha) \\ \downarrow \\ H_n(X^{k+1}, X^k) \cong \tilde{H}_n\left(X^{k+1}/X^k = \bigvee_{\alpha} S^{k+1} = \bigvee_{\alpha} \Delta^{k+1}/\partial\Delta^{k+1}\right) \\ \cong \bigoplus_{\alpha} \mathbb{Z} \end{array}$$

is an isom by the lemmas.

Everything now follows from

<u>Five Lemma</u>:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[r] \arrow[d, "\beta"'] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[r] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\circlearrowleft"] & D \arrow[r] \arrow[d, "\delta"'] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[d, "\varepsilon"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \end{tikzcd}$$

Pf: Diagram chase.

If the top and bottom are exact, and $\alpha, \beta, \delta, \varepsilon$ are $\cong$ then γ is also an $\cong$. ▨