# Lecture 26: Relative Homology



**Goal:** $\to \tilde{H}_n(A) \to \tilde{H}_n(X) \to \tilde{H}_n(X/A) \to \tilde{H}_{n-1}(A) \to$

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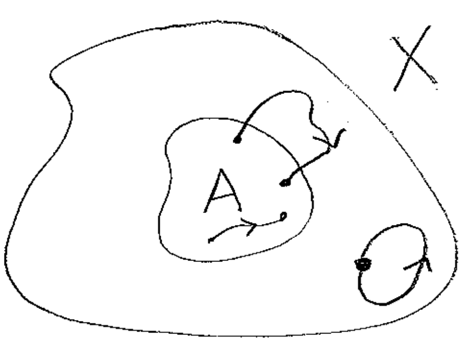
**Relative Homology:** (Stand-in for $H_n(X/A)$)




$$\begin{array}{ccc} C_n(X) & \xrightarrow{\partial} & C_{n-1}(X) \\ \cup\!| & & \cup\!| \\ C_n(A) & \xrightarrow{\partial} & C_{n-1}(A) \end{array}$$

**New complex:** $C_n(X,A) = C_n(X) / C_n(A)$ with induced $\partial$-map.

**Relative Homology:** $H_n(X,A)$ [= the homology of this chain complex]

**Relative cycles:** $c \in C_n(X)$ with $\partial c \in C_{n-1}(A)$

$c = 0$ in $H_n(X,A) \iff$ **relative boundary:**

$c = \partial d + a$ with $d \in C_{n+1}(X)$, $a \in C_n(A)$

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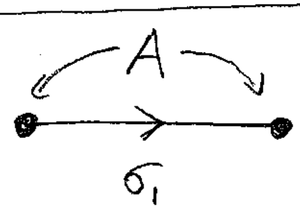
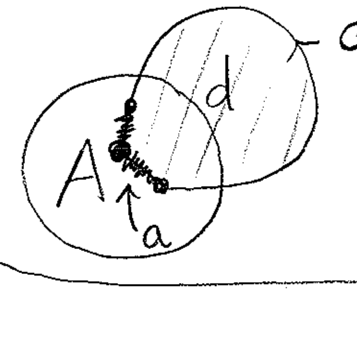
**Ex:**




$X = I \quad A = \partial I$

Compute using $\Delta$-complex homology.



$$C_2(X,A) \to C_1(X,A) \longrightarrow C_0(X,A) \longrightarrow 0$$

$$0 \qquad\qquad \mathbb{Z} = \langle \sigma_1 \rangle \qquad\qquad 0$$

$$\Rightarrow \quad H_n^{\Delta}(I, \partial I) = \begin{cases} \mathbb{Z} & n = 1 \\ 0 & \text{otherwise} \end{cases} \quad \left[ \cong \tilde{H}_n(I/_{\partial I} \cong S^1) \right]$$

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Have a long exact seq involving $H_n(X,A)$, as an instance of a general setup:

**Thm**: A short exact seq of chain complexes ($0 \to A_n \xrightarrow{i} B_n \xrightarrow{j} C_n \to 0$ is exact $\forall n$.)

$$0 \longrightarrow A_* \xrightarrow{i} B_* \xrightarrow{j} C_* \longrightarrow 0$$

where $i$ and $j$ are chain maps, gives rise to a long exact sequence:

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \xrightarrow{j_*}$$

**Ex**: $A_* = C_*(A) \quad B_* = C_*(X) \quad C_* = C_*(X,A)$.

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X,A) \to H_{n-1}(A) \to H_{n-1}(X) \to \cdots$$

Def. of $\partial : H_n(C) \to H_{n-1}(A)$:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\arrow[r] & A_{n+1} \arrow[r, "\partial"] \arrow[d, "i"] & A_n \arrow[r, "\partial"] \arrow[d, "i"] & A_{n-1} \arrow[r] \arrow[d, "i"] & {} \\
\arrow[r] & B_{n+1} \arrow[r] \arrow[d, "j"] & B_n \arrow[r] \arrow[d, "j"] & B_{n-1} \arrow[r] \arrow[d, "j"] & {} \\
\arrow[r] & C_{n+1} \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & {} \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

$$b \longmapsto \partial b, \qquad b \longmapsto c, \qquad a \longmapsto \partial b$$

Start with $[c] \in H_n(C)$. Pick $b \in B_n$ with $j(b) = c$.
Since $j(\partial b) = \partial(jb) = \partial c = 0$, $\exists\, a \in A_{n-1}$ with $i(a) = \partial b$.
Let $\partial[c] = [a]$, which is in $H_{n-1}(A)$ since

$$\partial a = 0 \iff 0 = i(\partial a) = \partial(i(a)) = \partial^2 b = 0.$$

Why is this well-defined?

- $a$ only dep on $b$, as $i$ is 1-1
- If $b'$ also has $j(b') = c$, $\exists\, d \in A_n$ with $i(d) = b' - b$

Thus

$$\begin{tikzcd}
d \arrow[r, mapsto] \arrow[d, mapsto] & a' - a \arrow[d, mapsto] \\
b' - b \arrow[r, mapsto] & \partial b' - \partial b
\end{tikzcd}$$

since $i$ is 1-1. So $[a'] = [a]$ in $H_{n-1}(A)$.

- If we replace $c$ with $c + \partial d$ where $d \in C_{n+1}$, then pick $b'' \in B_{n+1}$ with $j(b'') = d$. Then

$$\begin{tikzcd}
b'' \arrow[d, mapsto] & b' = b + \partial b'' \arrow[d, mapsto] \\
d & c' = c + \partial d
\end{tikzcd}$$

we can take $b' = b + \partial b''$ and so $\partial b = \partial b'$ and so this doesn't change anything.

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<u>Pf of Thm:</u>

<u>Exactness at $H_n(B)$:</u> $\boxed{\operatorname{im} i_* \subseteq \ker j_*}$ $j \circ i = 0$ at chain level

$\boxed{\ker j_* \subseteq \operatorname{im} i_*}$ $b \in B_n$ a cycle with $j_*([b]) = 0$, i.e. $j(b) = \partial c$. Pick $b' \in B_{n+1}$ with $j(b') = c$.

$$
\begin{tikzcd}
 & & 0 \arrow[d] & \\
 & & A_n \arrow[d] & \\
b' \arrow[ddr, bend right, mapsto] & B_{n+1} \arrow[r] \arrow[d, "j"] & B_n \arrow[d, "j"'] & b \arrow[dd, mapsto] \\
 & C_{n+1} \arrow[r] & C_n \arrow[dd] & \\
 & c \arrow[rr, mapsto] & & j(b) \\
 & & 0 &
\end{tikzcd}
$$

Then $b - \partial b'$ is in the kernel of $j$, so pick $a \in A_n$ with $i(a) = b - \partial b'$.

Now $[a] \in H_n(A)$ since

$$i(\partial a) = \partial(ia) = \partial b - \partial^2 b' = 0, \text{ and moreover}$$

$$i_*([a]) = [i(a)] = [b - \partial b'] = [b]. \qquad \blacksquare$$

Exactness at other locations are similar.

<u>Geometric interp. of $\partial$:</u>