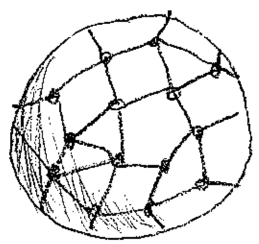
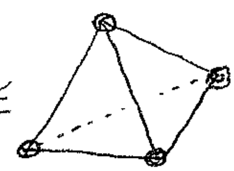
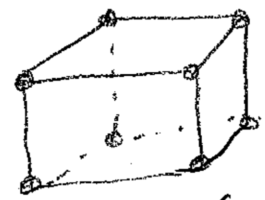
Lecture 33: Euler Characteristic

<u>Euler's Thm:</u> S^2 divided into polygons. Then



$$(\#\text{vert}) - (\#\text{edges}) + (\#\text{faces}) = 2$$

<u>Ex:</u> $S^2 \cong$  $\cong$ 

$4-6+4=2 \qquad 8-12+6=2$

[Easy application: Show there are only 5 regular solids.]

X a finite CW complex. Its <u>Euler Characteristic</u> is

$$\chi(X) = \sum (-1)^n (\#\text{ of } n\text{-cells})$$

<u>Thm:</u> If $X \simeq_{h.e.} Y$, then $\chi(X) = \chi(Y)$

[In particular, χ doesn't actually depend on the cell str. This gen. Euler's Thm.]

This follows from the below, since the RHS only depends on X.

<u>Thm:</u> $\chi(X) = \sum (-1)^n \operatorname{rank}(H_n(X))$

where rank(A - abelian gp) $= r$ if $A = \mathbb{Z}^r \oplus T$ ← elts of finite order.

Ex: $X = S^2$, $\sum (-1)^n \operatorname{rank}(H_n(X)) = 1 + 0 + 1 = 2$.

Key idea: Use that $H_n(X) \cong H_n^{CW}(X)$ to relate $\chi(X)$ to $H_*(X)$.

Consider the CW chains $C_n(X) = \mathbb{Z}^{k_n}$ (# of n-cells) with basis the n-cells e_α

$$\longrightarrow \mathbb{Z}^{k_3} \xrightarrow{d_3} \mathbb{Z}^{k_2} \xrightarrow{d_2} \mathbb{Z}^{k_1} \xrightarrow{d_1} \mathbb{Z}^{k_0} \longrightarrow 0$$

Since vector spaces are simpler than abelian gps replace $\mathbb{Z}$ with $\mathbb{Q}$ above;

$$\longrightarrow \mathbb{Q}^{k_3} \xrightarrow{d_3} \mathbb{Q}^{k_2} \xrightarrow{d_2} \mathbb{Q}^{k_1} \xrightarrow{d_1} \mathbb{Q}^{k_0} \longrightarrow 0$$

↑ basis consists of the 2-cells $= C_2(X;\mathbb{Q})$

where d_n is unchanged, in the sense that if $d_n(e_\alpha) = \sum_\beta c_\beta e_\beta$ this is still the case.
↑ integers

Set $H_n^{CW}(X;\mathbb{Q}) = \ker d_n / \operatorname{im} d_{n+1}$ which is a $\mathbb{Q}$-vector space.

Fact: $\operatorname{rank}(H_n^{CW}(X)) = \dim(H_n^{CW}(X;\mathbb{Q}))$

Rather than dwell on the (easy) proof of this fact, consider instead that we could have used $\mathbb{Q}$ instead of $\mathbb{Z}$ throughout the term.

E.g., there is a singular homology $H_n(X;\mathbb{Q}) \cong H_n^{CW}(X;\mathbb{Q})$ etc. We'll show

Thm: $\chi(X) = \sum(-1)^n \dim(H_n(X))$

which also gives invariance of χ.

Pf: Since $H_n(X;\mathbb{Q}) \cong H_n^{CW}(X;\mathbb{Q})$, this follows immediately from

Lemma: $0 \to C_m \xrightarrow{\partial_m} C_{m-1} \to \cdots \to C_1 \xrightarrow{\partial_1} C_0 \to 0$ be a chain-complex of finite dim'l vector spaces. Then
$$\sum_n (-1)^n \dim C_n = \sum_n (-1)^n \dim H_n$$

where $H_n = \ker \partial_n / \operatorname{im} \partial_{n+1}$ is the homology.

Proof: $\dim H_n = \dim(\ker \partial_n) - \dim(\operatorname{im} \partial_{n+1})$

$\dim C_n = \dim(\ker \partial_n) + \dim(\operatorname{im} \partial_n)$

Now mult by $(-1)^n$ and sum.

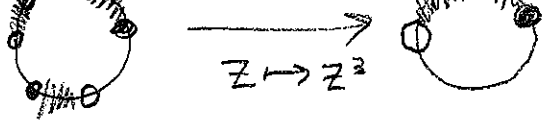
Ex: $\chi($  $) = 1 - 2g - 1 = 2 - 2g$

Ex: $\chi(S^n) = \begin{cases} 0 & n \text{ odd} \\ 2 & n \text{ even} \end{cases}$ by homology.

$\chi(\mathbb{R}P^n) = \begin{cases} 0 & n \text{ odd} \\ 1 & n \text{ even} \end{cases}$ since there's one cell in each dimension.

Thm: If $p: X \to Y$ is a covering map of finite degree n and Y is a CW complex, then $\chi(X) = n\,\chi(Y)$.

Pf: Pull back the CW str on Y to get one on X.



Last time, we proved

<u>Thm</u>: $G \neq 1$ is a finite group acting freely on S^n. If n is even, then $G \cong \mathbb{Z}/2\mathbb{Z}$.

<u>Alt Proof</u>: Let $M = S^n/G$. Then $S^n \to M$ is an $|G|$ cover, so $2 = \chi(S^n) = |G|\chi(M)$. As $\chi(M) \in \mathbb{Z}$, must have $|G| = 2$. ∎

<u>Underlying</u>: $M \simeq_{h.e.}$ a CW complex.

<u>Ex</u>:  can't cover 

$\chi = -2$ $\qquad\qquad$ $\chi = -4$.

If time remains,

talk more about homology with coefficients.