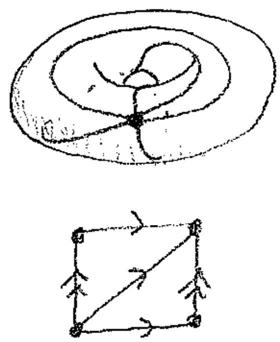
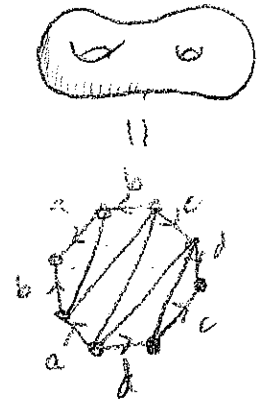
Lecture 32: Cellular Homology

Have simplical homology of a Δ-complex, but still gets complicated. A simplification is

6 cells



11 cells



<u>Cellular Homology</u>: X a CW complex.

$$f_\alpha : D^{n_\alpha} \longrightarrow X \quad , \quad f_\alpha : \partial D^{n_\alpha} \longrightarrow X^{n_\alpha - 1}$$

image a cell, attaching map.

Define
$$C_n^{CW}(X) = H_n(X^n, X^{n-1}) \cong \tilde{H}_n\left(X^n/X^{n-1} \cong \bigvee_\alpha S^n\right)$$
$$= \bigoplus_\alpha \mathbb{Z} \quad \text{where } \alpha \text{ indexes the } n\text{-cells.}$$

The boundary map $d_n : C_n^{CW}(X) \to C_{n-1}^{CW}(X)$ is $d_n = j_* \circ \partial$

$$\begin{tikzcd}
 & H_n(X^n) \arrow[d, "j_*"] & & \\
H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial"] \arrow[r, "d_{n+1}"'] & H_n(X^n, X^{n-1}) \arrow[rr, "d_n"'] \arrow[dr, "\partial"'] & & H_{n-1}(X^{n-1}, X^{n-2}) \\
 & & H_{n-1}(X^{n-1}) \arrow[ur, "j_*"'] &
\end{tikzcd}$$

where ∂, j_* are all from the long exact sequence of various pairs.

Note: $d_n \circ d_{n+1} = 0$ since

$$\begin{tikzcd} H_n(X^n) \arrow[dr, "j_*"] & & \\ & H_n(X^n, X^{n-1}) \arrow[dr, "\partial"] & \\ & & H_{n-1}(X^{n-1}) \end{tikzcd}$$

is from the long exact seq. of the pair, hence exact $\Rightarrow \partial \circ j_* = 0$.

Set $H_n^{CW}(X) = \ker d_n / \operatorname{im} d_{n+1}$.

Thm: $H_*^{CW}(X) \cong H_*(X)$ ← usual singular homology.

Pf: See Hatcher. Uses prob 3 (= Hatcher #22) from takehome midterm #2. ▨

Geometric meaning of d: For $n > 1$, consider the standard gen $i_n \in H_n(D^n, \partial D^n)$. Given a cell map $f_\alpha : D^n \to X$, take $e_\alpha = (f_\alpha)_*(i_n) \in C_n^{CW}(X) = \bigoplus_\alpha \mathbb{Z}$, which generates the α^{th} factor. Now

$$d_n(\alpha) = \sum_\beta c_\beta e_\beta \quad \text{where } \beta \text{ indexes the } (n-1)\text{-cells}$$

Idea: $c_\beta = \deg\Big(\partial D^n \xrightarrow{\Delta_{\alpha\beta}} \underbrace{X^{n-1} / X^{n-2} \vee \text{all } (n-1)\text{-cells except } \beta}_{Y = S^{n-1}_\beta}\Big)$

where $\Delta_{\alpha\beta} = g \circ f_\alpha$

f_α ← attaching map $\partial D^n \to X^{n-1}$

g ← quotient map.

<u>Reason</u>: $i_n \longmapsto e_\alpha \longmapsto \sum c_\beta e_\beta$

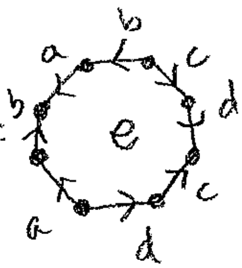
$$\begin{tikzcd}
H_n(D^n,\partial D^n) \arrow[r, "f_{\alpha *}"] \arrow[d, "\partial"'] \arrow[dr, phantom, "\circlearrowleft"] & H_n(X^n,X^{n-1}) \arrow[r, "d_n"] \arrow[d, "\partial"] \arrow[dr, phantom, "\circlearrowleft"] & H_{n-1}(X^{n-1},X^{n-2}) \arrow[d, "\cong"] \\
\tilde H_{n-1}(\partial D^n) \arrow[r, "f_{\alpha *}"] \arrow[drr, "\Delta_{\alpha\beta *}"'] & \tilde H_{n-1}(X^{n-1}) \arrow[ur, "j_*"] \arrow[dr, phantom, "\circlearrowleft"] & \tilde H_{n-1}\left(\bigvee_\beta S^{n-1}\right) \arrow[d, "\text{collapse all spheres except } \beta^{th}"] \\
 & & \tilde H_{n-1}(S^{n-1}_\beta)
\end{tikzcd}$$

$Z_{n-1} = \partial(i_n)$

standard gen

$\underbrace{c_\beta Z_{n-1}}$

image of $\sum c_\beta e_\beta$
$= \bar f_\alpha(Z_{n-1})$

<u>Ex</u>:  $=$ 

$= X$

$X^1 =$ 

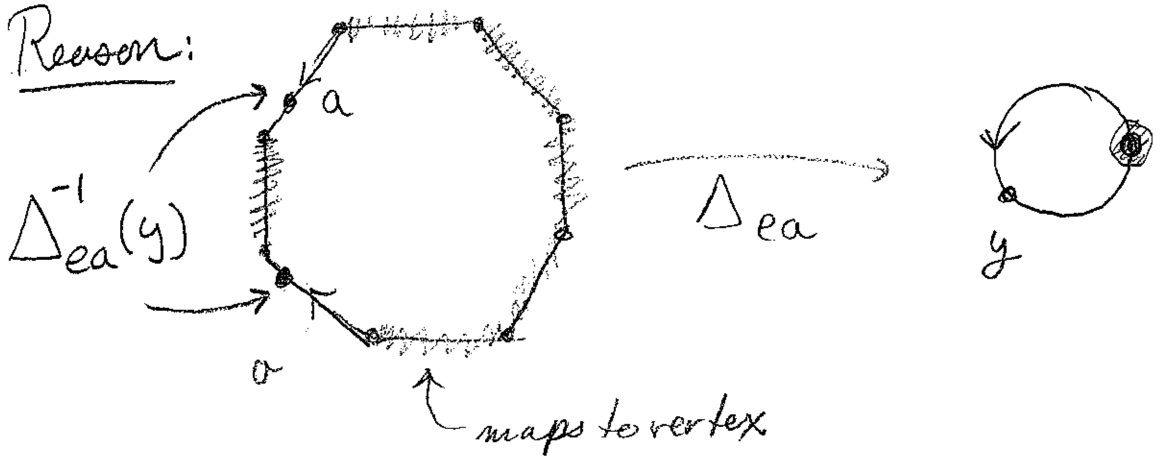
$$0 \longrightarrow \overset{\mathbb{Z}}{C_2^{CW}(X)} \longrightarrow \overset{\mathbb{Z}^4}{C_1^{CW}(X)} \longrightarrow \overset{\mathbb{Z}}{C_0^{CW}(X)} \longrightarrow 0$$

$d_1 = 0$; d_1 works the same as the usual simplicial ∂ map.

$d_2 = 0$ because the comp of $d_2(e)$ along a is

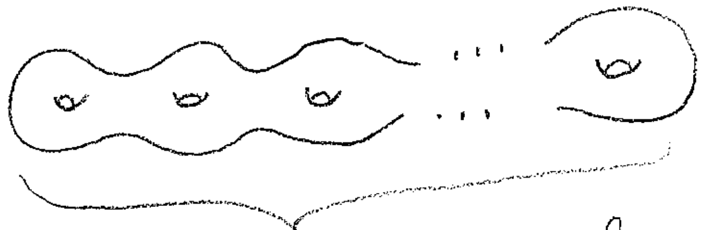
$\deg(\Delta_{ea} : S^1 \to X^1/_{b\cup c\cup d} =$  $) = 0$

Reason:

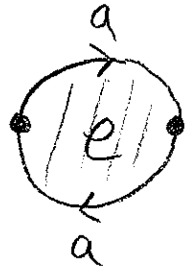


The two points of $\Delta_{ea}^{-1}(y)$ have opposite local degrees, so $\deg \Delta_{ea} = 0$.

So $H_n(X) \cong C_n^{CW}(X) = \begin{cases} \mathbb{Z} & n=2 \\ \mathbb{Z}^4 & n=1 \\ \mathbb{Z} & n=0 \end{cases}$

Similarly, if $X =$ 

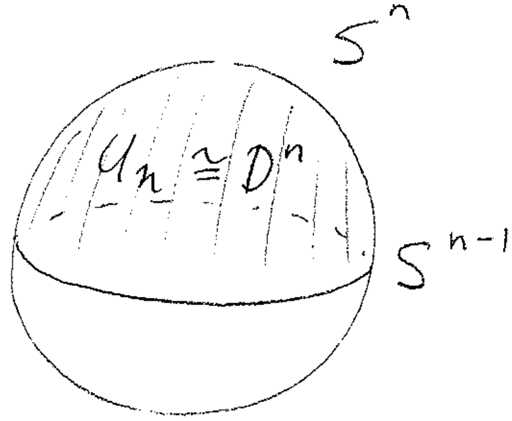
$H_n(X) = \begin{cases} \mathbb{Z} & n=2 \\ \mathbb{Z}^{2g} & n=1 \\ \mathbb{Z} & n=0 \end{cases}$ g holes, comes from a $4g$-gon

Ex: $\mathbb{RP}^2 =$ 

CW chains: $0 \longrightarrow \overset{2}{\mathbb{Z}} \xrightarrow[\times 2]{d_2} \overset{1}{\mathbb{Z}} \xrightarrow{d_1 = 0} \overset{0}{\mathbb{Z}} \longrightarrow 0$

$d_2(e) = 2a \qquad H_n(\mathbb{RP}^2) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & n=1 \\ \mathbb{Z} & n=0 \\ 0 & \text{otherwise} \end{cases}$

<u>Ex</u>: $\mathbb{R}P^n = S^n /_{x \mapsto -x}$



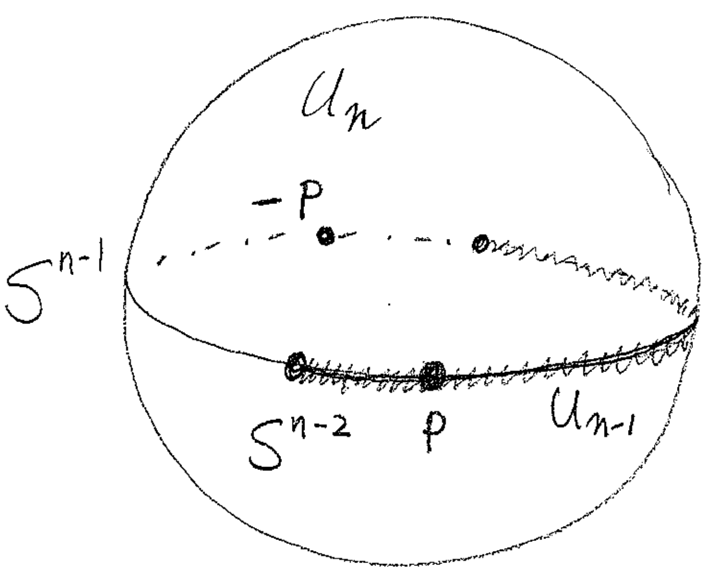
Let $q : S^n \to \mathbb{R}P^n$

$$\mathbb{R}P^n = \mathbb{R}P^{n-1} \cup q(U_n)$$

and $q|_{\mathrm{int}(U_n)}$ is 1-1. Inductively this gives a CW str to $\mathbb{R}P^n$, with 1 cell in each dim.

Computing the boundary map

$$\Delta : \partial U_n \to X^{n-1}/X^{n-2} = \mathbb{R}P^{n-1}/\mathbb{R}P^{n-2} \cong U_{n-1}/\partial U_{n-1} \cong S^{n-1}$$



Pick $\bar{p} \in \mathrm{int}(q(U_{n-1}))$

Then $\Delta^{-1}(\bar{p})$ consists of two points p and $-p$

$\deg_p \Delta = 1$

antipodal map

$$\deg_{-p} \Delta = \deg_{-p} \Delta \circ A$$
$$= (\deg_p \Delta) \deg_{-p} A = (-1)^{n+1}$$

<u>So</u> $d_n = 0$ when n is odd

$d_n = \times 2$ when n is even

Thus

$$H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} \text{ if } k = n \text{ and } n \text{ is odd} \\ \mathbb{Z}/2 \text{ if } k < n \text{ and } k \text{ is odd} \\ \mathbb{Z} \text{ if } k = 0 \\ 0 \text{ otherwise} \end{cases}$$

Since

n odd

$$0 \longrightarrow \overset{n}{\mathbb{Z}} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \rightarrow \cdots$$

n even

$$0 \longrightarrow \overset{n-1}{\mathbb{Z}} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times 2} \cdots$$