Lecture 30: The meaning of degree.

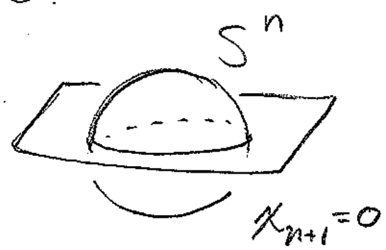
Last time:

Degree: $f: S^n \to S^n$, $H_n(S^n) \to H_n(S^n)$

$(\deg f) \in \mathbb{Z}$ $\quad$ $\underset{\alpha}{\text{gen'ly}} \longmapsto (\deg f)\alpha$

Basic facts: $\deg(id_{S^n}) = 1$ $\quad \deg\left(\begin{smallmatrix} f \text{ not} \\ \text{onto} \end{smallmatrix}\right) = 0$.

$\deg\left(\begin{smallmatrix}\text{reflection} \\ \text{in plane through } \vec{0}\end{smallmatrix}\right) = -1$



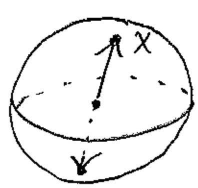
Start here:

⑥ $\deg(f \circ g) = (\deg f)(\deg(g))$ since

$$H_n(S^n) \xrightarrow{g_*} H_n(S^n) \xrightarrow{f_*} H_n(S^n)$$

$$\alpha \longmapsto (\deg g)\alpha \longmapsto (\deg f)\cdot(\deg g)\cdot\alpha$$

⑦ A = antipodal map, then $\deg(A) = (-1)^{n+1}$



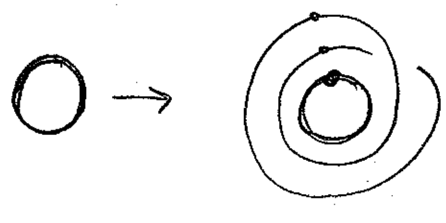
$x \longmapsto -x$

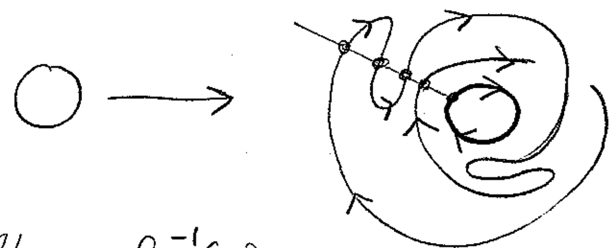
since A = comp of $n+1$ reflections.

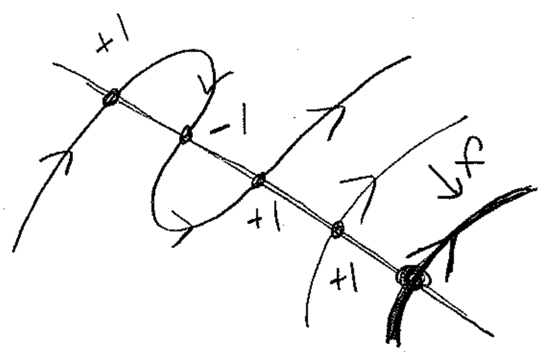
Q: What does $\deg f$ measure?

A: Roughly, $\deg f$ = # of pts in $f^{-1}(p)$ "counted with signs."

Ex: $z \mapsto z^2$ deg $= 2$ A homotopic map f

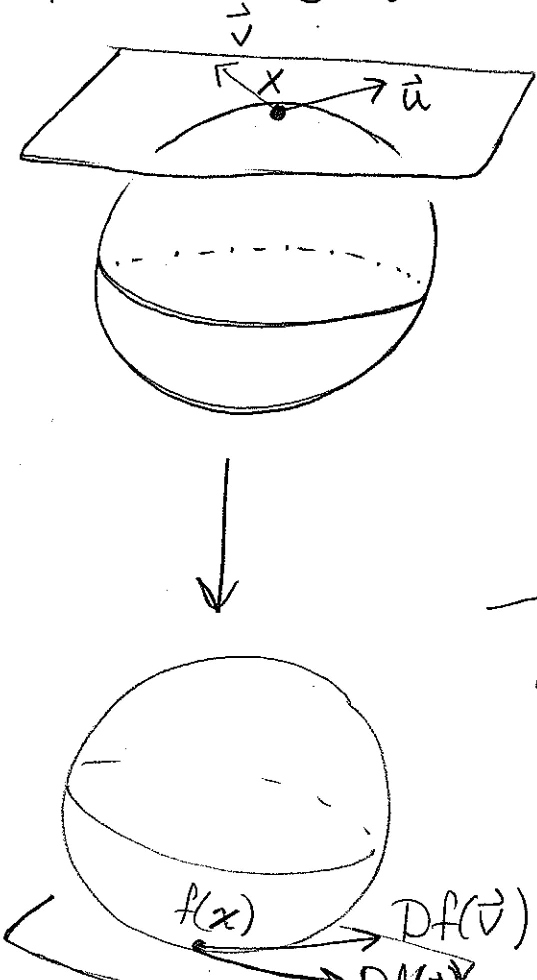






Here, $f^{-1}(p)$ consists of 2, 3 or 4 points. But if we count ± 1 depending on whether the orient. agree, then these numbers typically sum to 2.

$f: S^n \to S^n$ smooth [e.g. the restriction of a smooth fn on some nbhd of S^n]



$$T_x S^n = \{ v \in \mathbb{R}^{n+1} \mid v \cdot x = 0 \}$$

the tangent space.

$$T_x S^n \xrightarrow{Df \text{ a linear map}} T_{f(x)} S^n$$

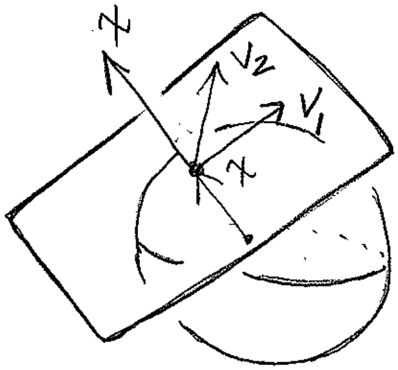
$\boxed{\text{Q: What are the reg. values in the } S^1 \text{ exs?}}$

Def: $y \in S^n$ is a regular value if $Df: T_x S^n \to T_y S^n$ is an isomorphism for all $x \in f^{-1}(y)$

($\Rightarrow f$ is a diffeo near x)

<u>Sard's Thm</u>: The set of regular values is dense in S^n. [and has full Lebesgue measure.]

$T_x S^n$ has an orient induced by $\{v_1, \ldots, v_n\}$ is a pos. basis iff $\det\begin{pmatrix} | & | & & | & | \\ v_1 & v_2 & \cdots & v_n & x \\ | & | & & | & | \end{pmatrix} > 0$.

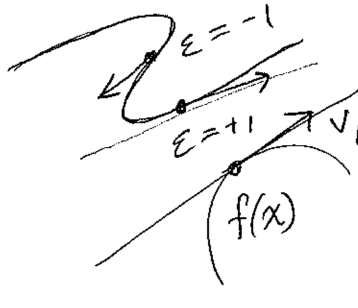


If $Df: T_x(S^n) \to T_{f(x)}(S^n)$ is an isom, set

$$\varepsilon(x) = \begin{cases} +1 & \text{if } Df \text{ pres. orient} \\ -1 & \text{if } Df \text{ rev. orient} \end{cases}$$

<u>Thm</u>: Suppose $f: S^n \to S^n$ is smooth. If $y \in S^n$ is a reg. value,

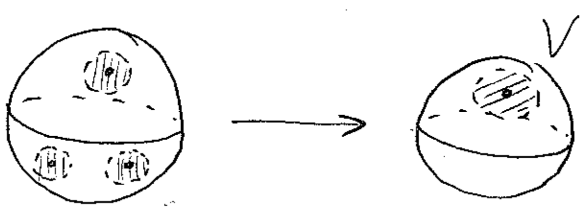
$$\deg f = \sum_{x \in f^{-1}(y)} \varepsilon(x)$$



<u>Cor</u>: $\# f^{-1}(y) \equiv (\deg f) \bmod 2$ if y is a reg. value.

[To define ε for any map & pt, will use local homology.]

Local degrees: Suppose $y \in S^n$ has $f^{-1}(y) = \{x_i\}$ finite.
Choose a nbhd V of y and disjoint nbhds U_i of x_i with $f(U_i) \subseteq V$.



Here $\bar{\alpha}_i$ comes from the gen α of $H_n(S^n)$.

$\mathbb{Z}$ gen by $\bar{\alpha}_i$ $\cong$ $H_n(U_i, U_i \setminus x_i)$ $\qquad$ $\mathbb{Z}$ gen by $\bar{\alpha}$ $\cong$ $H_n(V, V \setminus y)$

$$\begin{tikzcd}
 & H_n(U_i, U_i \setminus x_i) \arrow[r, "f_*"] \arrow[dl, "i_*", "\text{excision } \cong"'] \arrow[d, "k_i"] & H_n(V, V \setminus y) \arrow[d, "\cong"] \\
H_n(S^n, S^n \setminus x_i) & H_n(S^n, S^n \setminus f^{-1}(y)) \arrow[l, "p_i"'] \arrow[r, "f_*"] & H_n(S^n, S^n \setminus y) \\
 & H_n(S^n) \arrow[u, "j_*"] \arrow[ul, "j_*", "\cong"'] \arrow[r, "f_*"] & H_n(S^n) \arrow[u, "\cong"']
\end{tikzcd}$$

$\bar{\alpha} \longmapsto (\deg_{x_i} f)\bar{\alpha}$

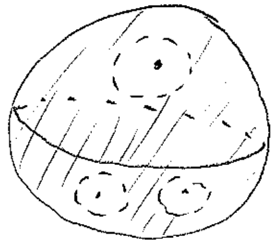
$H_n(S^n, S^n \setminus x_i)$ gen by $\bar{\alpha}_i'$; $H_n(S^n, S^n \setminus y)$: $\bar{\alpha}$

gen by $\alpha \longmapsto (\deg f)\,\alpha$

If $f|_{U_i}$ is a homeo to V, then $\deg_{x_i} f = \pm 1$.
For a smooth map, $\deg_{x_i} f = \varepsilon(x_i)$. [On HW.]

Thm: $\deg f = \sum \deg_{x_i} f$

Pf: Excising $S^n \setminus \bigcup U_i$,



we see

$$\begin{tikzcd} H_n(S^n, S^n \setminus f^{-1}(y)) & \bigoplus H_n(U_i, U_i \setminus x_i) \arrow[l, "\oplus k_i"', "\cong"] \arrow[d, "\cong"] \\ & \bigoplus H_n(S^n, S^n \setminus x_i) \end{tikzcd}$$

gen by $\bar{\alpha}_i \longmapsto$ gen by $\bar{\alpha}_i'$

Note: $j_*(\alpha) = \sum' \bar{\alpha}_i$.

Also, $\bar{\alpha}_i \in H_n(S^n, S^n \setminus f^{-1}(y))$

$$\bar{\alpha}_i \overset{f_*}{\longmapsto} (\deg_{x_i} f)\,\bar{\alpha}$$

So $f_* \circ j_*(\alpha) = \left(\sum \deg_{x_i} f\right)\bar{\alpha} = (\deg f)\,\bar{\alpha}$ ▨

App: Given $k \in \mathbb{Z}$, $\exists f : S^n \to S^n$ of deg k.

Pf: Consider

$$\begin{tikzcd} S^n \arrow[r] \arrow[rr, bend right, "f"'] & \bigvee_k S^n \arrow[r] & S^n \end{tikzcd}$$

where $S^n \to \bigvee_k S^n$ is gotten by collapsing $S^n \setminus (k \text{ disjoint balls})$.