Lecture 37:

Goal: [Note: integer coeffs.

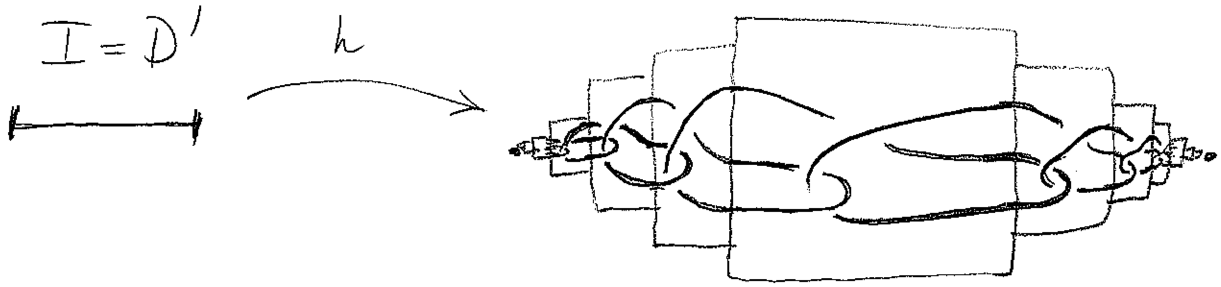
Thm: a) $h: D^k \hookrightarrow S^n$, then $\tilde{H}_i(S^k \setminus h(D^k)) = 0$ for all i.

b) $h: S^k \hookrightarrow S^n$ with $k < n$.

Then $\tilde{H}_i(S^n \setminus h(S^k)) = \begin{cases} \mathbb{Z} & \text{if } i = n-k-1 \\ 0 & \text{otherwise} \end{cases}$

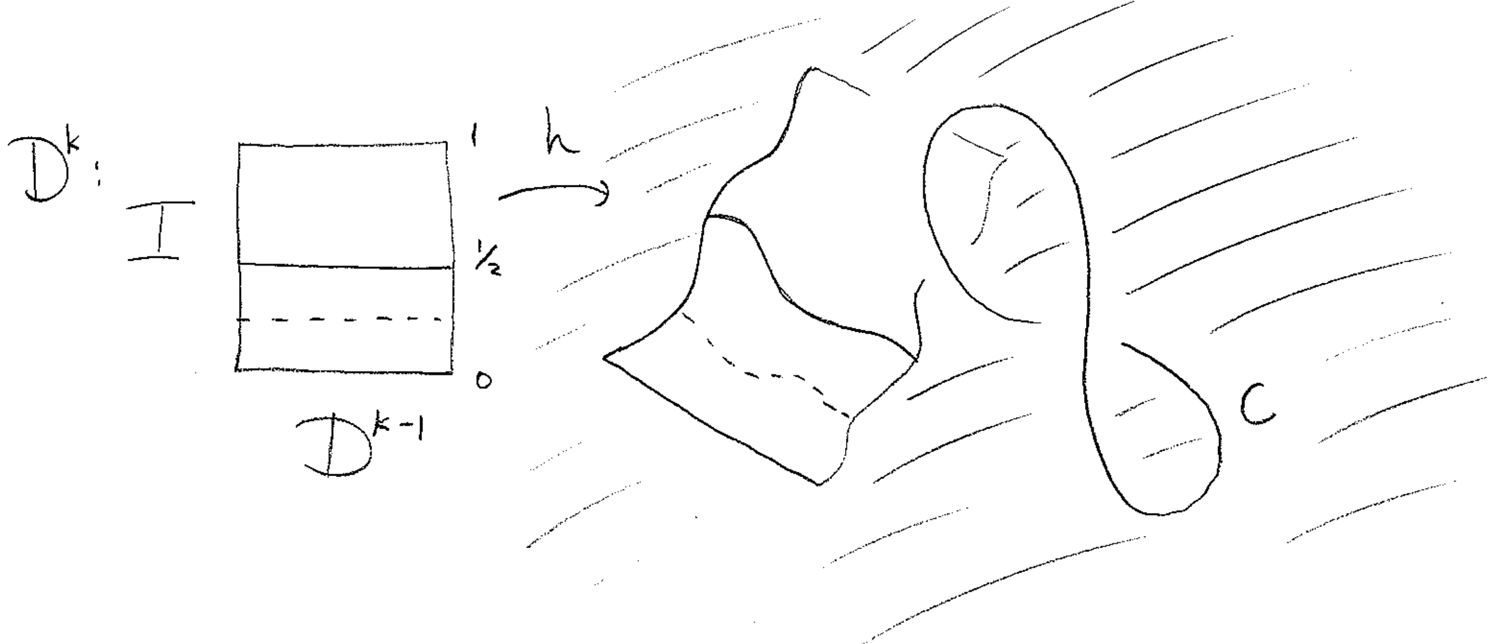
Cor: If S^n is embedded in S^{n+1}, then the complement has two (path) components.

Reminder: h can be very far from standard



Pf: ⓐ Induct on k with n fixed. As $D^0 = \{pt\}$ the base case is clear. Set $D = h(D^k)$. Suppose $\alpha \in \tilde{H}_i(S^n \setminus D)$ is $\neq 0$, and fix $c \in C_i(S^n \setminus D)$ rep α.

[Idea: $D^k = D^{k-1} \times I$. Shrink final factor to reduce to earlier case.]



Set $A = S^n \setminus h(D^{k-1} \times [0, 1/2])$ [Trying to isolate the problem.]

$B = S^n \setminus h(D^{k-1} \times [1/2, 1])$

Thus $A \cap B = S^n \setminus D$ and $A \cup B = S^n \setminus h(D^{k-1} \times \{1/2\})$

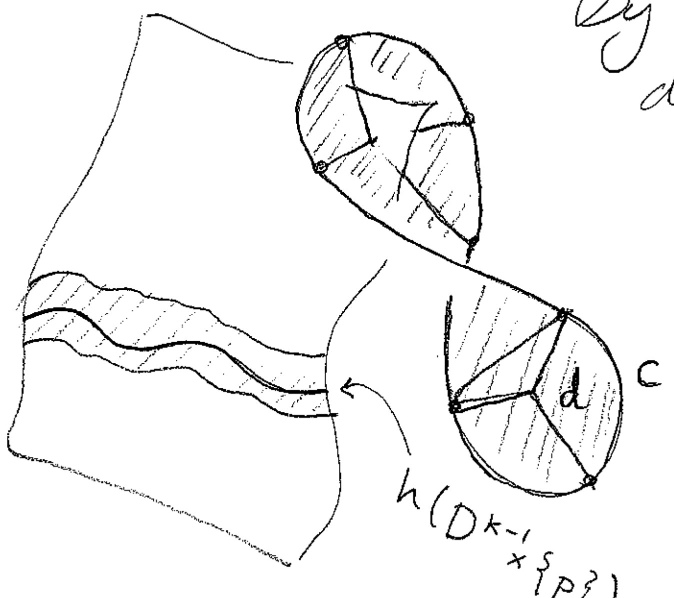
By M-V have

$$\underbrace{\tilde{H}_{i+1}(A \cup B)}_{\overset{\shortparallel}{0}} \to \underset{\alpha}{\tilde{H}_i(A \cap B)} \xrightarrow[\cong]{i_* \oplus -i_*} \tilde{H}_i(A) \oplus \tilde{H}_i(B) \to \underbrace{\tilde{H}_i(A \cup B)}_{\overset{\shortparallel}{0} \text{ by induct.}}$$

So $i_*(\alpha) \neq 0$ in one of $\tilde{H}_i(A)$ and $\tilde{H}_i(B)$. Repeating, can construct nested intervals I_j of length 2^{-j} s.t. $\alpha \neq 0$ in each $\tilde{H}_i(S^n \setminus h(D^{k-1} \times I_j))$.

Let $p = \cap I_j$. By induction $\alpha = 0$ in $\tilde{H}_i(S^n \setminus h(D^{k-1} \times \{p\}))$.

Let $d \in C_i(S^n \setminus h(D^{k-1} \times \{p\}))$ be such that $\partial d = c$.



By compactness, $\exists j$ s.t. d is also disjoint from $h(D^{k-1} \times I_j)$

But then $\alpha = 0$ in

$$\tilde{H}_i(S^n \setminus h(D^{k-1} \times I_j))$$

a contradiction.

So $\tilde{H}_i(S^n \setminus D) = 0 \ \forall i$.

(b) Again, induct on k. When $k = 0$, $S^n \setminus \{\text{two pts}\} \cong S^{n-1} \times \mathbb{R}$ and we're done. In general, $S^k = U \cup L$ where $U, L \cong D^k$ meeting in S^{k-1}.

Let $A = S^n \setminus h(U)$, $B = S^n \setminus h(L)$ and so

$$A \cap B = S^n \setminus h(S^k) \qquad A \cup B = S^n \setminus h(S^{k-1})$$

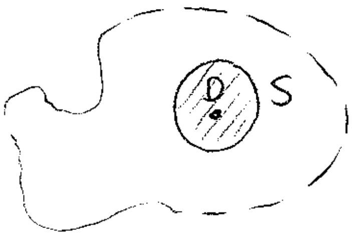
Then M-V gives

$$\underbrace{\tilde{H}_{i+1}(A) \oplus \tilde{H}_{i+1}(B)}_{0} \to \tilde{H}_{i+1}(A \cup B) \xrightarrow{\cong} \tilde{H}_i(A \cap B) \to \underbrace{\tilde{H}_i(A) \oplus H_i(B)}_{0}$$

as needed. $\blacksquare$

Invariance of Domain: Suppose $X \subseteq \mathbb{R}^n$ is homeo to an open set in $\mathbb{R}^n$. Then X is open.

Pf: Can replace $\mathbb{R}^n$ with S^n. Any $x \in X$ is contained in $D \cong D^n$, and let S corresp to ∂D^n under this homeo. Now $S^n \setminus D$ is open and connected by Thm (a). [open sets in S^n are conn $\Leftrightarrow$ path conn.]



$X \subseteq S^n$

Also $S^n \setminus S$ is open and has two comp. by Thm (b). Then

$$S^n \setminus S = \underbrace{S^n \setminus D}_{\text{conn}} \amalg \underbrace{D \setminus S}_{\text{conn as an open ball.}}$$

$\Rightarrow$ these are the two comp, hence open in S^n.

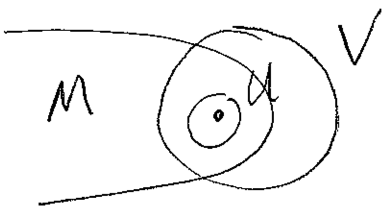
So $D \setminus S$ is an open nbhd of $x \subseteq X$, so X is open. ▨

Cor: M a cpt n-mfld, N a connected n-mfld. Then any embedding $M \hookrightarrow N$ is onto.

Cor^2: S^n does not embed in $\mathbb{R}^n$.

Cor^3: $\mathbb{R}^n$ does not contain a subspace $\cong$ to $\mathbb{R}^k$ for $k > n$. Pf: If it did, $\mathbb{R}^n \supseteq S^n$. ▨

Pf of Cor: M is closed in N as it is cpt. and N is Hausdorff. Each $x \in M$ has a nbhd $U \overset{\text{open}}{\subseteq} M$ and $V \overset{\text{open}}{\subseteq} N$ which is homeo to $\mathbb{R}^n$.



Can assume $U \subseteq V \overset{\text{Thm}}{\Rightarrow} U$ is open in $V \Rightarrow U$ is open in N. Then M open + closed in $N \Rightarrow M = N$. $\blacksquare$

If time remains:

Thm: (Mazur Brown 60s) $f: S^n \hookrightarrow S^{n+1}$ is locally flat, i.e. extends to an embedding $S^n \times I \hookrightarrow S^n$, then $\exists h: S^{n+1} \to S^{n+1}$ with $h(f(S^n)) = S^n$.