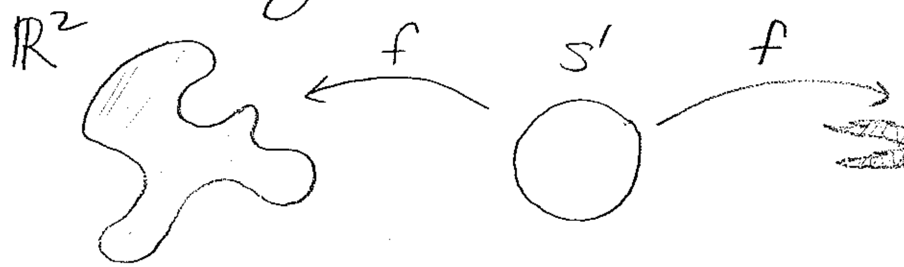
Lecture 36: Jordan curve thm + friends

Jordan Curve Thm (1887)

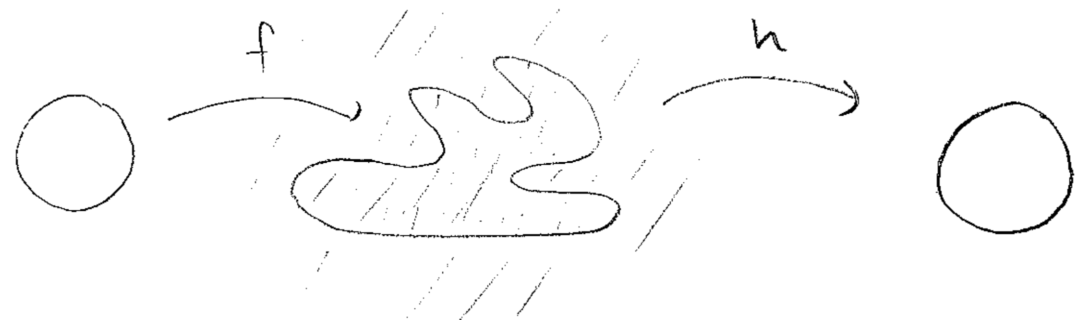
[1st correct proof by Veblen in 1905.]

$f: S^1 \hookrightarrow \mathbb{R}^2$ an embedding. Then $\mathbb{R}^2 \setminus f(S^1)$ has exactly two connected components.



Schoenflies Thm (1906):

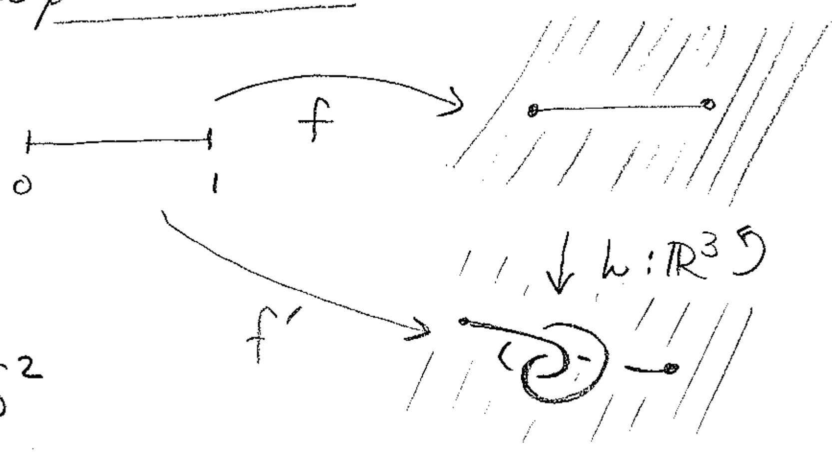
$f: S^1 \hookrightarrow \mathbb{R}^2$ an embedding. Then $\exists\, h: \mathbb{R}^2 \circlearrowleft \cong$ s.t. $h \circ f(S^1) = S^1$ and $h \circ f|_{S^1} = id$.



J.C.T. generalizes to all $S^n \hookrightarrow \mathbb{R}^{n+1}$, but Schoenflies doesn't!

Wild arcs and spheres in $\mathbb{R}^3$:

Standard:

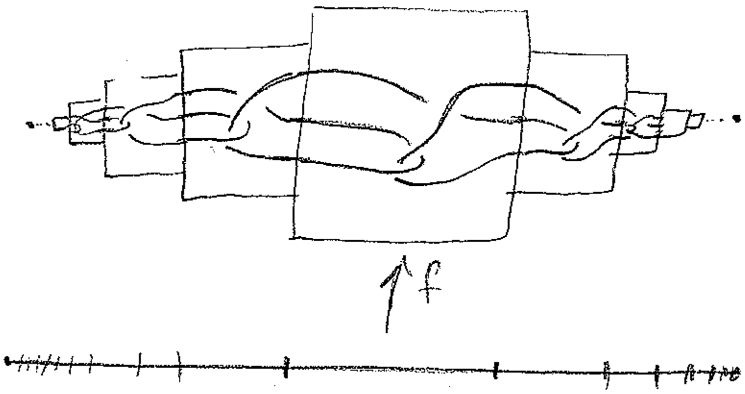


$\mathbb{R}^3 \setminus f(I) \simeq_{h.e.} S^2$

Wild: Each box is the same

This is a cont map $f: I \to \mathbb{R}^3$

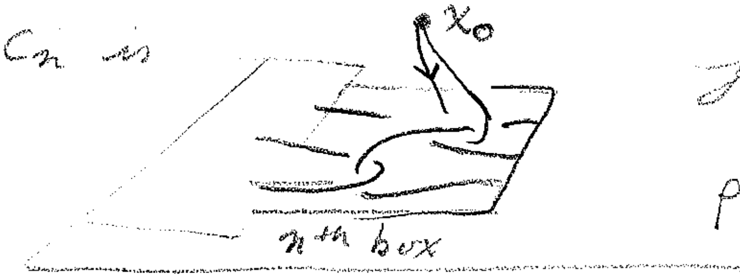
[only poss issue is at the endpoints.]



A little geometric calc with Van Kampen's Thm gives

$$\pi_1(\mathbb{R}^3 \setminus f(I)) = \left\langle \{c_n\}_{n\in\mathbb{Z}} \mid c_{n-1}c_nc_{n+1} = c_nc_{n+1}c_{n-1}c_n \right\rangle$$

where c_n is



This group is nontrivial

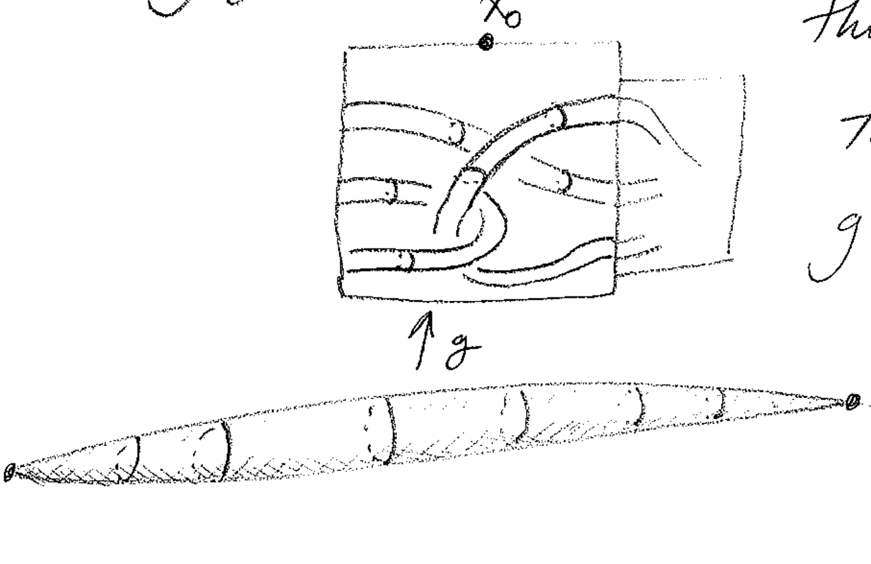
$$\rho: \pi_1 \longrightarrow S_5$$

$$c_n \longmapsto \begin{cases} (12345) & n \text{ odd} \\ (14235) & n \text{ even} \end{cases}$$

[So no homeo of $\mathbb{R}^3$ taking $f(I)$ to a standard one.]

Thickening gives a map $g: S^2 \hookrightarrow \mathbb{R}^3$ where S^2 gets thinner as we go toward the ends.

g is an embedding (homeo onto image)



Then $\pi_1(\mathbb{R}^3 \setminus g(S^2), x_0) \neq 1$,
so $\nexists$ a homeo h of $\mathbb{R}^3$ taking $g(S^2)$ to the round sphere.

Consider $S^{n-1} \hookrightarrow S^n$ [for symmetry.]

Cor: $f: S^{n-1} \hookrightarrow S^n$ an embedding.
Then $S^n \setminus f(S^{n-1})$ has two connected components.

Thm: a) $h: D^k \hookrightarrow S^n$. Then $\tilde{H}_i(S^n \setminus h(D^k)) = 0$ for all i.

b) $h: S^k \hookrightarrow S^n$ with $k < n$ then

$$\tilde{H}_i(S^n \setminus h(S^k)) = \begin{cases} \mathbb{Z} & \text{if } i = n-k-1 \\ 0 & \text{otherwise} \end{cases}$$

[Explain why the cor follows from (b).]

[Point: While $S^n \setminus h(D^k)$ can vary, its homology can't.]

[Will prove next time.]

Meyer-Vietoris: $X = \text{int}(A) \cup \text{int}(B)$. Have an exact seq:

$$\to H_n(A\cap B) \xrightarrow[\varphi]{} H_n(A) \oplus H_n(B) \xrightarrow[\psi]{} H_n(X) \xrightarrow[\partial]{} H_{n-1}(A \cap B) \to$$

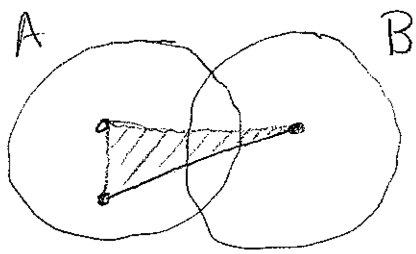
where $\varphi(c) = (i_*(c), -i_*(c))$ and $\psi(a,b) = i_*(a) + i_*(b)$.

[Equivalent to the long exact seq of the pair.]

$$x \longmapsto (x, -x)$$

Pf: $0 \to C_n(A\cap B) \to C_n(A) \oplus C_n(B) \to C_n(A+B) \longrightarrow 0$

$$a, b \longmapsto a+b$$

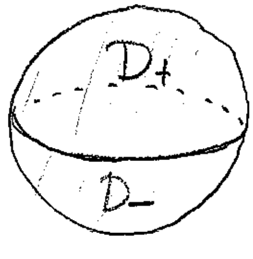


sums of chains in A + chains in B.

Excision says that $H_*(C_n(A+B)) \cong H_*(X)$.

Also works for $X = A \cup B$ where A, B are def. retracts of nbhds U, V with $U \cap V$ def retracting to $A \cap B$. Eg. X a CW complex, A, B subcomplexes.

Ex: $S^n = D^n_+ \cup D^n_-$



$A \cap B = S^{n-1}$

$$\underbrace{\tilde{H}_k(D^n_+) \oplus \tilde{H}_k(D^n_-)}_{=0} \longrightarrow \tilde{H}_k(S^n) \xrightarrow{\cong} \tilde{H}_{k-1}(S^{n-1}) \longrightarrow \tilde{H}_{k-1}(D^n_+) \oplus \tilde{H}_k(D^n_-) = 0$$