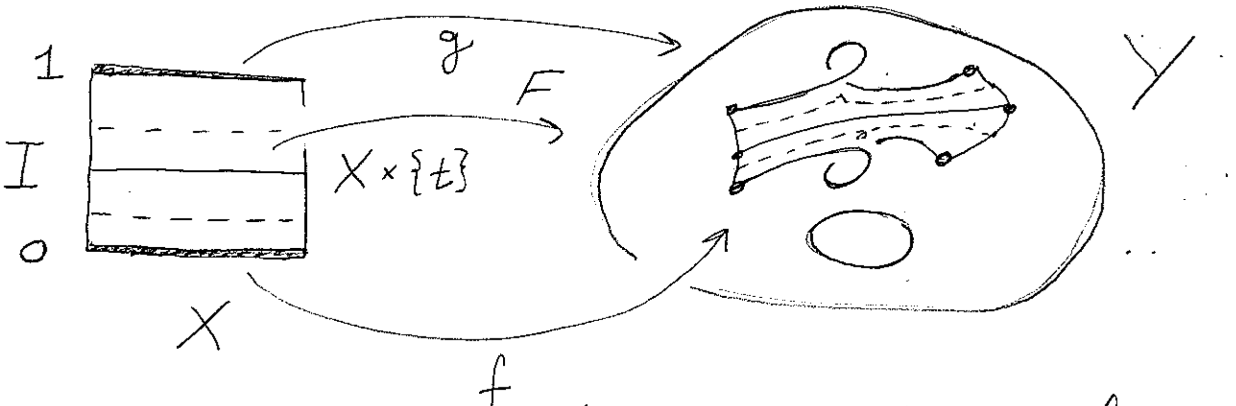
Lecture 3:

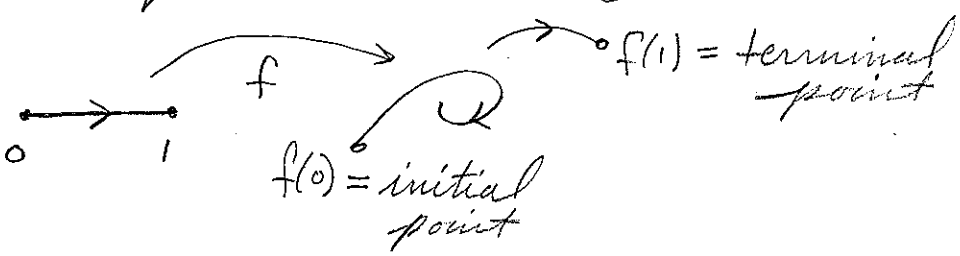
[Start with a def that's key for the whole course.]

Def: Cont. maps $f, g : X \to Y$ are homotopic ($f \simeq g$) if $\exists$ a cont map $F : X \times \underset{= [0,1]}{I} \to Y$ with

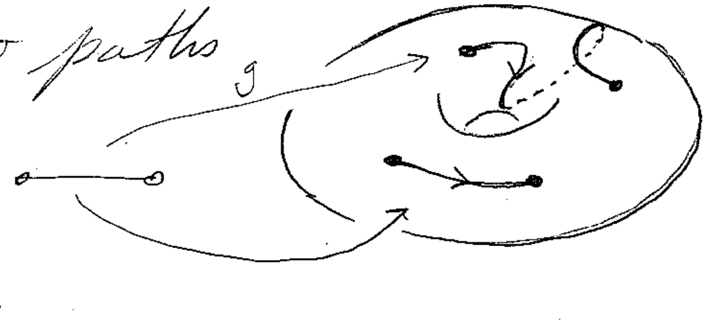
$F(x,0) = f(x)$ and $F(x,1) = g(x)$ for all $x \in X$.



Def: A path in Y is just a map $f : I \to Y$



Query: Are these two paths homotopic?

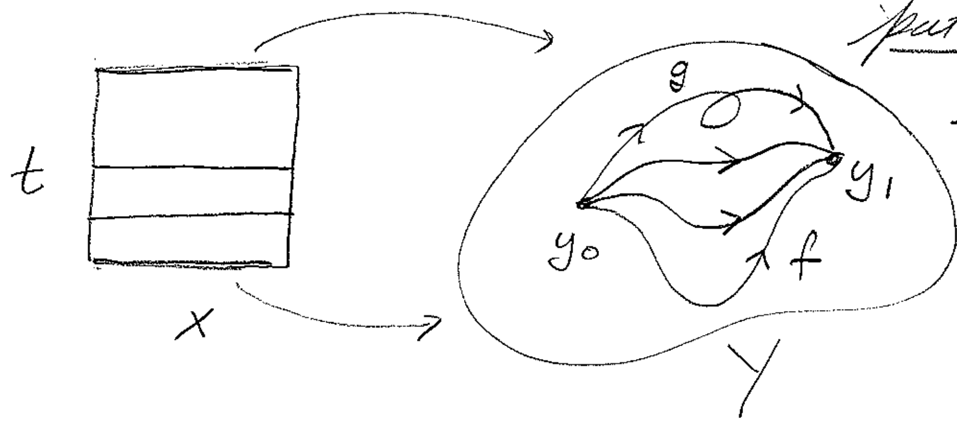


Ans: Yes for any space that is path connected.

$\uparrow$ do people know this?

[So we need to refine the notion for paths.]

Def: Paths $f, g : I \to Y$ going from y_0 to y_1 are <u>path homotopic</u> ($f \simeq_p g$) if $\exists F : I \times I \to Y$ such that:



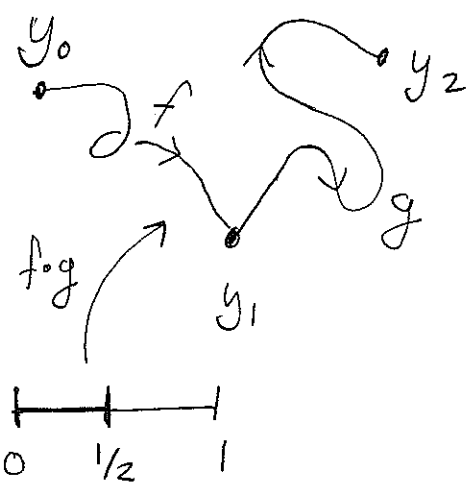
$F(x,0) = f(x)$ $\forall x \in I$
$F(x,1) = g(x)$

<u>and</u>

$F(0,t) = y_0$ $\forall t \in I$
$F(1,t) = y_1$

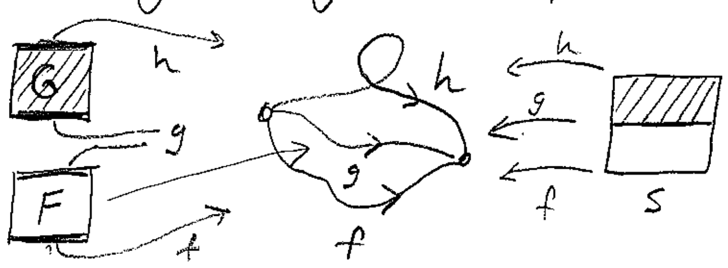
[Remember, I'm trying to create a group out of loops/paths...]

<u>Concatenation</u>: f path from y_0 to y_1, g a path from y_1 to y_2. The composit path from y_0 to y_2 is



$$f \cdot g(s) = \begin{cases} f(2s) & \text{for } s \in [0, 1/2] \\ g(2s-1) & \text{for } s \in [1/2, 1] \end{cases}$$

<u>Note</u>: $\simeq_p$ is an equivalence relation on paths from y_0 to y_1: $f \simeq_p g$ and $g \simeq_p h \Rightarrow f \simeq_p h$

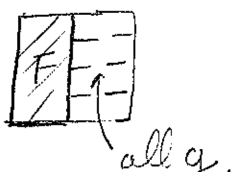


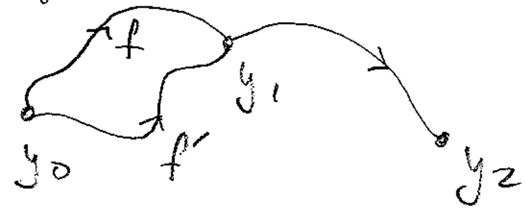
$$H(s,t) = \begin{cases} F(s, 2t) & \text{for } t \in [0, 1/2] \\ G(s, 2t-1) & \text{for } t \in [1/2, 1] \end{cases}$$

Use $[f]$ to denote the path homotopy class of f. Note that the end pts of $[f]$ make sense. If $[f]$ goes from y_0 to y_1 and $[g]$ goes from y_1 to y_2, then set

$$[f]\cdot[g] = [f\cdot g]$$

which is well-defined since if $f' \simeq_p f$ via $\boxed{F}$ have $f'\cdot g \simeq_p f\cdot g$ via

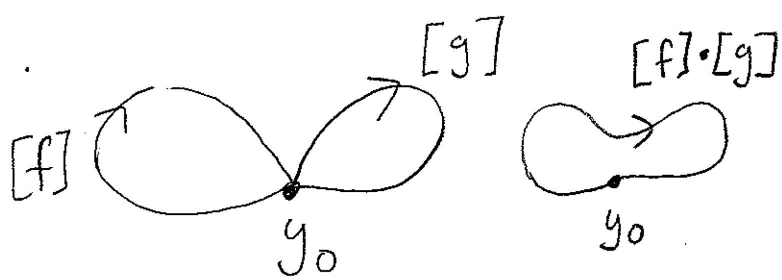




Def: Y top space, $y_0 \in Y$. The <u>fundamental group</u> of Y based at y_0 is

$$\pi_1(Y, y_0) = \left\{ [f] \;\middle|\; \begin{array}{l} f \text{ starts and ends} \\ \text{at } y_0 \end{array} \right\}$$

with the operation $\cdot$.



<u>Check</u> [Group axioms]

<u>Assoc</u>: $([f]\cdot[g])\cdot[h] = [f]\cdot([g]\cdot[h])$

<u>Ident</u>: $1 = [\text{constant path at } y_0]$

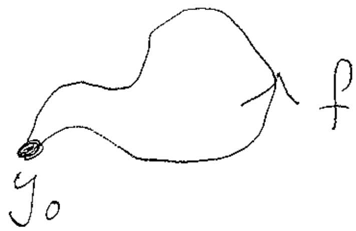
<u>Inverses</u>: $[f]^{-1} = [\bar{f}]$ where $\bar{f}(s) = f(1-s)$

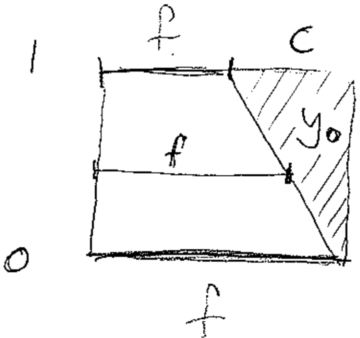
Proof:

<u>Ident</u>: Let c be the const path at y_0,

$[f] \in \pi_1(Y, y_0)$. Want $[f]\cdot[c] = [c]\cdot[f] = [f]$

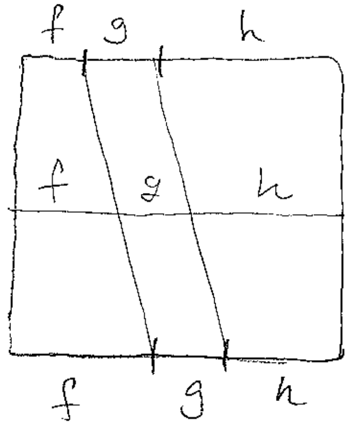
<u>Note</u>: $f\cdot c \neq c\cdot f \neq f$.



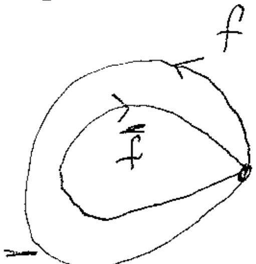


$$F(s,t) = \begin{cases} f\left(\frac{s}{1-t/2}\right) & \text{if } s \le \frac{1}{2}t \\ y_0 & \text{otherwise.} \end{cases}$$

<u>Assoc</u>:



<u>Inverses</u>:



$[f]\cdot[\bar{f}]$