Lecture 2: The invariants of algebraic topology

Algebraic Topology: $X \rightsquigarrow F(X)$ alg. object, e.g. a group

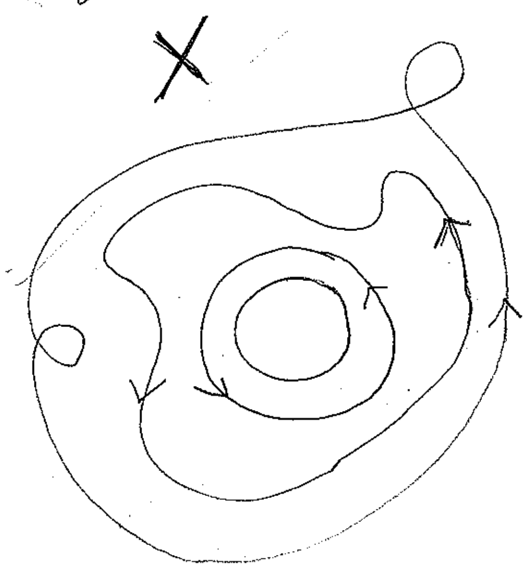
X: top space

Main Invariants: [depends only on the homeomorphism type]

Fundamental Group [Ch 1 of Hatcher]

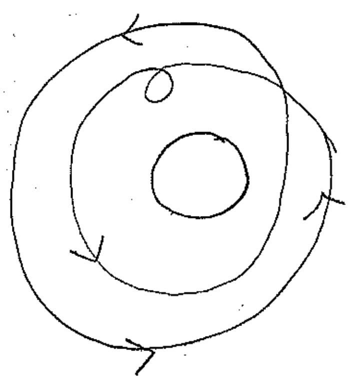
$\pi_1 X =$ "The set of loops in X up to continuous deformation."

Ex: $X = \mathbb{R}^2 \setminus B_1(\vec{0})$



Three loops that are all the same in $\pi_1 X$:

A loop that's different

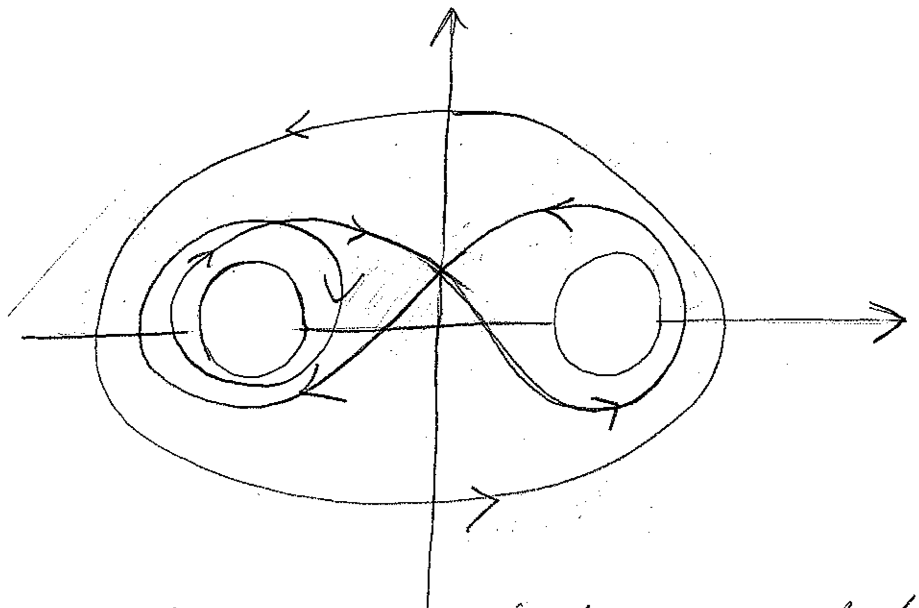


Formally, a loop is a (continuous) map from

$(S^1 = \text{circle} = O) \longrightarrow X$

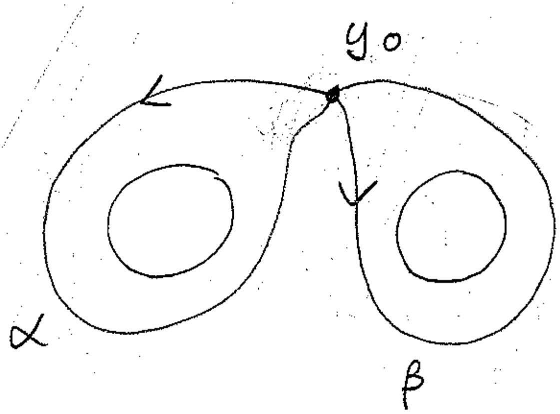
In this case, there's a $\mathbb{Z}$'s worth of loops. (c.f. winding # in complex anal.)

$\underline{Ex}$: $Y = \mathbb{R}^2 \setminus \left(B_1((2,0)) \cup B_1((-2,0)) \right)$

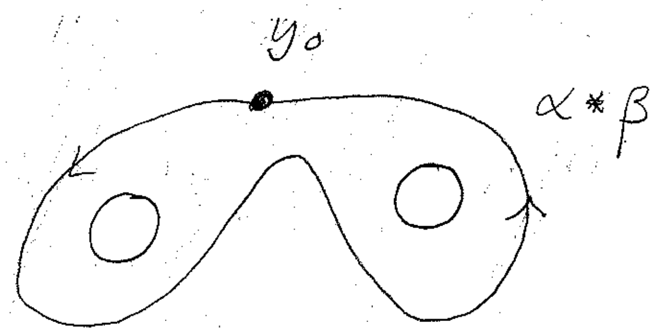


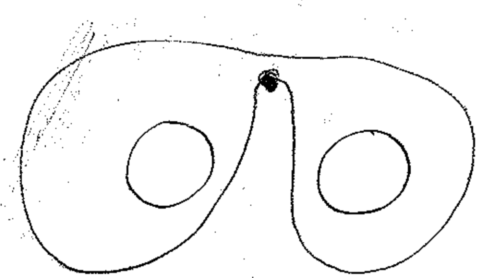
$\underline{Note}$: $\pi_1 X$ and $\pi_1 Y$ are both countably infinite. [To tell them apart, need to make π_1 into a group.]

Fix a point y_0 in Y. [the basepoint.] and just consider loops that start at y_0



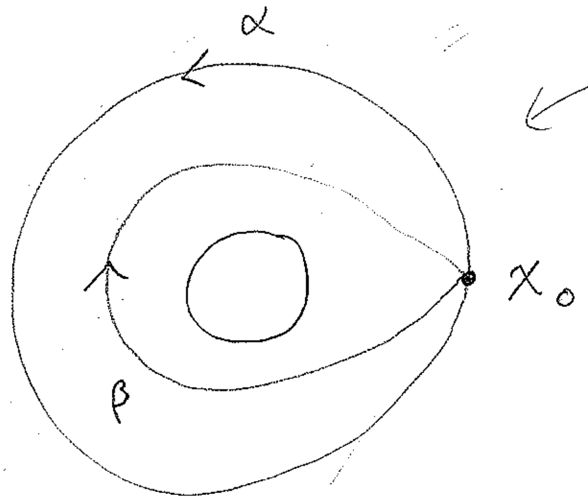
Group opperation: concatenation



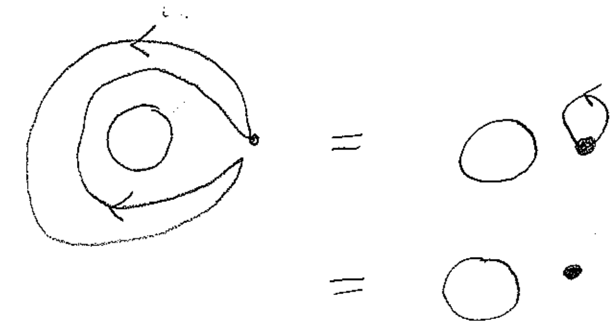


$\beta * \alpha \neq \alpha * \beta$.

For X, notice that if



then $\alpha * \beta$ is



that is, $\beta = \alpha^{-1}$! Turns out that

$$\pi_1 X = \{\alpha^n \mid n \in \mathbb{Z}\} \xrightarrow{\cong} (\mathbb{Z}, n)$$

$$\text{loop} \longmapsto \text{winding number}$$

In contrast, $\pi_1 Y$ is not commutative (see last page), so $\pi_1 Y \not\cong \pi_1 X$ and $Y \not\cong X$.

<u>Higher Homotopy Groups</u> [Math 526; Ch 4 of Hatcher]

$$S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$$

 , $S^3 = \mathbb{R}^3 \cup \{\infty\}$, etc...

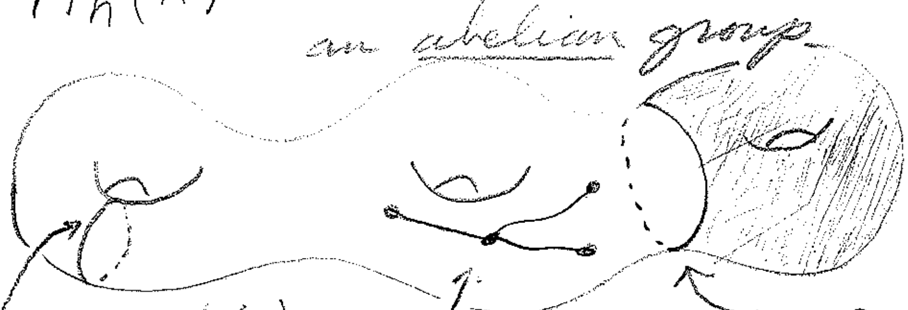
$$\pi_n(X) = \{S^n \to X\} \quad \text{[up to deformation]}$$

For $n > 1$, this is an abelian group; very powerful, capturing a huge amount of info about X. Really hard to compute, e.g. $\pi_n(S^2 = \ominus)$ are not all known.

Homology: [Ch 2 of Hatcher]

$H_n(X) =$ "n-dim'l things w/o boundary / boundaries on n+1 dim'l things"

an abelian group



[Easy to compute, but still very useful.]

Cohomology: [Math 526; Ch 3 of Hatcher]

$H^n(X)$ - "dual" notion to homology.

[When X is a smooth manifold, can be defined using differential forms.]

Has $H^n(X) \times H^m(X) \longrightarrow H^{n+m}(X)$, making $\bigoplus H^n(X)$ into an algebra. We will cover (co)homology from a topological viewpoint, but they are an important tool in many areas of math.