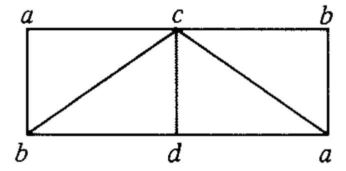

Figure 3.11

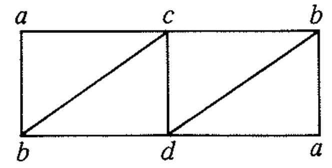

Figure 3.12

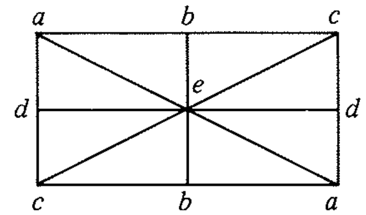

Figure 3.13

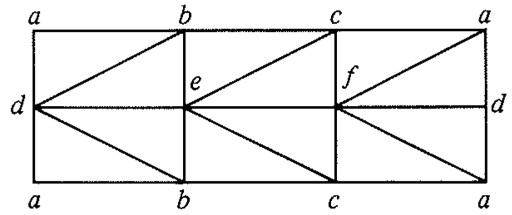

Figure 3.14

2. Describe the spaces determined by the labelled complexes in Figures 3.11–3.14.
3. Prove Lemma 3.2.
4. Let S be a set with a partial order relation $\leq$. A standard technique in combinatorics is to associate with S the abstract complex $\mathcal{S}$ whose vertices are the elements of S and whose simplices are the finite simply-ordered subsets of S. Suppose one is given the partial order on $\{a_1, \ldots, a_8\}$ generated by the following relations:

$$a_1 \leq a_3 \leq a_7 \leq a_8; \qquad a_1 \leq a_5 \leq a_7;$$
$$a_2 \leq a_6 \leq a_8; \qquad a_2 \leq a_5.$$

Describe a geometric realization of $\mathcal{S}$.

§4. REVIEW OF ABELIAN GROUPS

In this section, we review some results from algebra that we shall be using—specifically, facts about abelian groups.

We write abelian groups additively. Then 0 denotes the neutral element, and $-g$ denotes the additive inverse of g. If n is a positive integer, then ng denotes the n-fold sum $g + \cdots + g$, and $(-n)g$ denotes $n(-g)$.

We denote the group of integers by $\mathbf{Z}$, the rationals by $\mathbf{Q}$, and the complex numbers by $\mathbf{C}$.

Homomorphisms

If $f : G \to H$ is a homomorphism, the **kernel** of f is the subgroup $f^{-1}(0)$ of G, the **image** of f is the subgroup $f(G)$ of H, and the **cokernel** of f is the quotient group $H/f(G)$. We denote these groups by $\ker f$ and $\operatorname{im} f$ and $\operatorname{cok} f$, respectively. The map f is a monomorphism if and only if the kernel of f vanishes (i.e., equals the trivial group). And f is an epimorphism if and only if the cokernel of f vanishes; in this case, f induces an isomorphism $G/\ker f \simeq H$.

Free abelian groups

An abelian group G is **free** if it has a **basis**—that is, if there is a family $\{g_\alpha\}_{\alpha \in J}$ of elements of G such that each $g \in G$ can be written uniquely as a finite sum

$$g = \Sigma n_\alpha g_\alpha,$$

with n_α an integer. Uniqueness implies that each element g_α has infinite order; that is, g_α generates an infinite cyclic subgroup of G.

More generally, if each $g \in G$ can be written as a finite sum $g = \Sigma n_\alpha g_\alpha$, but not necessarily uniquely, then we say that the family $\{g_\alpha\}$ **generates** G. In particular, if the set $\{g_\alpha\}$ is finite, we say that G is **finitely generated.**

If G is free and has a basis consisting of n elements, say $g_1, \ldots, g_n$, then it is easy to see that every basis for G consists of precisely n elements. For the group $G/2G$ consists of all cosets of the form

$$(\Sigma \epsilon_i g_i) + 2G,$$

where $\epsilon_i = 0$ or 1; this fact implies that the group $G/2G$ consists of precisely 2^n elements. The number of elements in a basis for G is called the **rank** of G.

It is true more generally that if G has an infinite basis, any two bases for G have the same cardinality. We shall not use this fact.

A crucial property of free abelian groups is the following: If G has a basis $\{g_\alpha\}$, then any function f from the set $\{g_\alpha\}$ to an abelian group H extends uniquely to a homomorphism of G into H.

One specific way of constructing free abelian groups is the following: Given a set S, we define the **free abelian group G generated by S** to be the set of all functions $\phi : S \to \mathbf{Z}$ such that $\phi(x) \neq 0$ for only finitely many values of x; we add two such functions by adding their values. Given $x \in S$, there is a characteristic function ϕ_x for x, defined by setting

$$\phi_x(y) = \begin{cases} 0 & \text{if} \quad y \neq x, \\ 1 & \text{if} \quad y = x. \end{cases}$$

The functions $\{\phi_x \mid x \in S\}$ form a basis for G, for each function $\phi \in G$ can be written uniquely as a finite sum

$$\phi = \Sigma n_x \phi_x,$$

where $n_x = \phi(x)$ and the summation extends over all x for which $\phi(x) \neq 0$. We often abuse notation and identify the element $x \in S$ with its characteristic function ϕ_x. With this notation, the general element of G can be written uniquely as a finite "formal linear combination"

$$\phi = \Sigma n_\alpha x_\alpha$$

of the elements of the set S.

If G is an abelian group, an element g of G has **finite order** if $ng = 0$ for some positive integer n. The set of all elements of finite order in G is a subgroup T of G, called the **torsion subgroup.** If T vanishes, we say G is **torsion-free.** A free abelian group is necessarily torsion-free, but not conversely.

If T consists of only finitely many elements, then the number of elements in T is called the **order** of T. If T has finite order, then each element of T has finite order; but not conversely.

Internal direct sums

Suppose G is an abelian group, and suppose $\{G_\alpha\}_{\alpha \in J}$ is a collection of subgroups of G, indexed bijectively by some index set J. Suppose that each g in G can be written uniquely as a finite sum $g = \Sigma g_\alpha$, where $g_\alpha \in G_\alpha$ for each α. Then G is said to be the **internal direct sum** of the groups G_α, and we write

$$G = \oplus_{\alpha \in J} G_\alpha.$$

If the collection $\{G_\alpha\}$ is finite, say $\{G_\alpha\} = \{G_1, \ldots, G_n\}$, we also write this direct sum in the form $G = G_1 \oplus \cdots \oplus G_n$.

If each g in G can be written as a finite sum $g = \Sigma g_\alpha$, but not necessarily uniquely, we say simply that G is the **sum** of the groups $\{G_\alpha\}$, and we write $G = \Sigma G_\alpha$, or, in the finite case, $G = G_1 + \cdots + G_n$. In this situation, we also say that the groups $\{G_\alpha\}$ **generate** G.

If $G = \Sigma G_\alpha$, then this sum is direct if and only if the equation $0 = \Sigma g_\alpha$ implies that $g_\alpha = 0$ for each α. This in turn occurs if and only if for each fixed index α_0, one has

$$G_{\alpha_0} \cap \left(\sum_{\alpha \neq \alpha_0} G_\alpha \right) = \{0\}.$$

In particular, if $G = G_1 + G_2$, then this sum is direct if and only if $G_1 \cap G_2 = \{0\}$.

The resemblance to free abelian groups is strong. Indeed, if G is free with basis $\{g_\alpha\}$, then G is the direct sum of the subgroups $\{G_\alpha\}$, where G_α is the infinite cyclic group generated by g_α. Conversely, if G is a direct sum of infinite cyclic subgroups, then G is a free abelian group.

If G is the direct sum of subgroups $\{G_\alpha\}$, and if for each α, one has a homomorphism f_α of G_α into the abelian group H, the homomorphisms $\{f_\alpha\}$ extend uniquely to a homomorphism of G into H.

Here is a useful criterion for showing G is a direct sum:

Lemma 4.1. *Let G be an abelian group. If G is the direct sum of the subgroups $\{G_\alpha\}$, then there are homomorphisms*

$$j_\beta : G_\beta \to G \qquad \text{and} \qquad \pi_\beta : G \to G_\beta$$

such that $\pi_\beta \circ j_\alpha$ is the zero homomorphism if $\alpha \neq \beta$, and the identity homomorphism if $\alpha = \beta$.

Conversely, suppose $\{G_\alpha\}$ is a family of abelian groups, and there are homomorphisms j_β and π_β as above. Then j_β is a monomorphism. Furthermore, if the groups $j_\alpha(G_\alpha)$ generate G, then G is their direct sum.

Proof. Suppose $G = \oplus G_\alpha$. We define j_β to be the inclusion homomorphism. To define π_β, write $g = \Sigma g_\alpha$, where $g_\alpha \in G_\alpha$ for each α; and let $\pi_\beta(g) = g_\beta$. Uniqueness of the representation of g shows π_β is a well-defined homomorphism.

Consider the converse. Because $\pi_\alpha \circ j_\alpha$ is the identity, j_α is injective (and π_α is surjective). If the groups $j_\alpha(G_\alpha)$ generate G, every element of G can be written as a finite sum $\Sigma j_\alpha(g_\alpha)$, by hypothesis. To show this representation is unique, suppose

$$\Sigma j_\alpha(g_\alpha) = \Sigma j_\alpha(g'_\alpha).$$

Applying π_β, we see that $g_\beta = g'_\beta$. □

Direct products and external direct sums

Let $\{G_\alpha\}_{\alpha \in J}$ be an indexed family of abelian groups. Their **direct product** $\Pi_{\alpha \in J} G_\alpha$ is the group whose underlying set is the cartesian product of the sets G_α, and whose group operation is component-wise addition. Their **external direct sum** G is the subgroup of the direct product consisting of all tuples $(g_\alpha)_{\alpha \in J}$ such that $g_\alpha = 0_\alpha$ for all but finitely many values of α. (Here 0_α is the zero element of G_α.) The group G is sometimes also called the "weak direct product" of the groups G_α.

The relation between internal and external direct sums is described as follows: Suppose G is the external direct sum of the groups $\{G_\alpha\}$. Then for each β, we define $\pi_\beta : G \to G_\beta$ to be projection onto the βth factor. And we define $j_\beta : G_\beta \to G$ by letting it carry the element $g \in G_\beta$ to the tuple $(g_\alpha)_{\alpha \in J}$, where $g_\alpha = 0_\alpha$ for all α different from β, and $g_\beta = g$. Then $\pi_\beta \circ j_\alpha = 0$ for $\alpha \neq \beta$, and $\pi_\alpha \circ j_\alpha$ is the identity. It follows that G equals the *internal* direct sum of the groups $G'_\alpha = j_\alpha(G_\alpha)$, where G'_α is isomorphic to G_α.

Thus the notions of internal and external direct sums are closely related. The difference is mainly one of notation. For this reason, we customarily use the notations

$$G = G_1 \oplus \cdots \oplus G_n \qquad \text{and} \qquad G = \oplus G_\alpha$$

to denote either internal or external direct sums, relying on the context to make clear which is meant (if indeed, it is important). With this notation, one can for instance express the fact that G is free abelian of rank 3 merely by writing $G \simeq \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$.

If G_1 is a subgroup of G, we say that G_1 is a **direct summand** in G if there is a subgroup G_2 of G such that $G = G_1 \oplus G_2$. In this case, if H_i is a subgroup of G_i, for $i = 1,2$, then the sum $H_1 + H_2$ is direct, and furthermore,

$$\frac{G}{H_1 \oplus H_2} \simeq \frac{G_1}{H_1} \oplus \frac{G_2}{H_2}.$$

In particular, if $G = G_1 \oplus G_2$, then $G/G_1 \simeq G_2$.

Of course, one can have $G/G_1 \simeq G_2$ without its following that $G = G_1 \oplus G_2$; that is, G_1 may be a subgroup of G without being a direct summand in G. For instance, the subgroup $n\mathbf{Z}$ of the integers is not a direct summand in $\mathbf{Z}$, for that would mean that

$$\mathbf{Z} \simeq n\mathbf{Z} \oplus G_2$$

for some subgroup G_2 of $\mathbf{Z}$. But then G_2 is isomorphic to $\mathbf{Z}/n\mathbf{Z}$, which is a group of finite order, while no subgroup of $\mathbf{Z}$ has finite order.

Incidentally, we shall denote the group $\mathbf{Z}/n\mathbf{Z}$ of integers modulo n simply by $\mathbf{Z}/n$, in accordance with current usage.

The fundamental theorem of finitely generated abelian groups

There are actually two theorems that are important to us. The first is a theorem about subgroups of free abelian groups. We state it here, and give a proof in §11:

Theorem 4.2. *Let F be a free abelian group. If R is a subgroup of F, then R is also a free abelian group. If F has rank n, then R has rank $r \leq n$; furthermore, there is a basis $e_1, \ldots, e_n$ for F and integers $t_1, \ldots, t_k$ with $t_i > 1$ such that*

(1) *$t_1 e_1, \ldots, t_k e_k, e_{k+1}, \ldots, e_r$ is a basis for R.*

(2) *$t_1 \mid t_2 \mid \cdots \mid t_k$, that is, t_i divides t_{i+1} for all i.*

The integers $t_1, \ldots, t_k$ are uniquely determined by F and R, although the basis $e_1, \ldots, e_n$ is not.

An immediate corollary of this theorem is the following:

Theorem 4.3 (The fundamental theorem of finitely generated abelian groups). *Let G be a finitely generated abelian group. Let T be its torsion subgroup.*

(a) *There is a free abelian subgroup H of G having finite rank β such that $G = H \oplus T$.*

(b) *There are finite cyclic groups $T_1, \ldots, T_k$, where T_i has order $t_i > 1$, such that $t_1 \mid t_2 \mid \cdots \mid t_k$ and*

$$T = T_1 \oplus \cdots \oplus T_k.$$

(c) *The numbers β and $t_1, \ldots, t_k$ are uniquely determined by G.*

The number β is called the **betti number** of G; the numbers $t_1, \ldots, t_k$ are called the **torsion coefficients** of G. Note that β is the rank of the free abelian group $G/T \simeq H$. The rank of the subgroup H and the orders of the subgroups T_i are uniquely determined, but the subgroups themselves are not.

Proof. Let S be a finite set of generators $\{g_i\}$ for G; let F be the free abelian group on the set S. The map carrying each g_i to itself extends to a homomorphism carrying F *onto* G. Let R be the kernel of this homomorphism. Then $F/R \simeq G$. Choose bases for F and R as in Theorem 4.2. Then

$$F = F_1 \oplus \cdots \oplus F_n$$

where F_i is infinite cyclic with generator e_i; and

$$R = t_1F_1 \oplus \cdots \oplus t_kF_k \oplus F_{k+1} \oplus \cdots \oplus F_r.$$

We compute the quotient group as follows:

$$F/R \simeq (F_1/t_1F_1 \oplus \cdots \oplus F_k/t_kF_k) \oplus (F_{r+1} \oplus \cdots \oplus F_n).$$

Thus there is an isomorphism

$$f: G \rightarrow (\mathbf{Z}/t_1 \oplus \cdots \oplus \mathbf{Z}/t_k) \oplus (\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}).$$

The torsion subgroup T of G must be mapped to the subgroup $\mathbf{Z}/t_1 \oplus \cdots \oplus \mathbf{Z}/t_k$ by f, since any isomorphism preserves torsion subgroups. Parts (a) and (b) of the theorem follow. Part (c) is left to the exercises. □

This theorem shows that any finitely generated abelian group G can be written as a finite direct sum of cyclic groups; that is,

$$G \simeq (\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}) \oplus \mathbf{Z}/t_1 \oplus \cdots \oplus \mathbf{Z}/t_k.$$

with $t_i > 1$ and $t_1 \mid t_2 \mid \cdots \mid t_k$. This representation is in some sense a "canonical form" for G. There is another such canonical form, derived as follows:

Recall first the fact that if m and n are relatively prime positive integers, then

$$\mathbf{Z}/m \oplus \mathbf{Z}/n \simeq \mathbf{Z}/mn.$$

It follows that any finite cyclic group can be written as a direct sum of cyclic groups whose orders are powers of primes. Theorem 4.3 then implies that for any finitely generated group G,

$$G \simeq (\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}) \oplus (\mathbf{Z}/a_1 \oplus \cdots \oplus \mathbf{Z}/a_s)$$

where each a_i is a power of a prime. This is another canonical form for G, since the numbers a_i are uniquely determined by G (up to a rearrangement), as we shall see. The numbers a_i are called the **invariant factors** of G.

EXERCISES

1. Show that if G is a finitely generated abelian group, every subgroup of G is finitely generated. (This result does not hold for non-abelian groups.)
2. (a) Show that if G is free, then G is torsion-free.
   (b) Show that if G is finitely generated and torsion-free, then G is free.

(c) Show that the additive group of rationals $\mathbf{Q}$ is torsion-free but not free. [*Hint:* If $\{g_\alpha\}$ is a basis for $\mathbf{Q}$, let β be fixed and express $g_\beta/2$ in terms of this basis.]

3. (a) Show that if m and n are relatively prime, then $\mathbf{Z}/m \oplus \mathbf{Z}/n$ is cyclic of order mn.
   (b) If $G \simeq \mathbf{Z}/18 \oplus \mathbf{Z}/36$, express G as a direct sum of cyclic groups of prime power order.
   (c) If $G \simeq \mathbf{Z}/2 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/3 \oplus \mathbf{Z}/3 \oplus \mathbf{Z}/9$, find the torsion coefficients of G.
   (d) If $G \simeq \mathbf{Z}/15 \oplus \mathbf{Z}/20 \oplus \mathbf{Z}/18$, find the invariant factors and the torsion coefficients of G.

4. (a) Let p be prime; let $b_1, \ldots, b_k$ be non-negative integers. Show that if

$$G \simeq (\mathbf{Z}/p)^{b_1} \oplus (\mathbf{Z}/p^2)^{b_2} \oplus \cdots \oplus (\mathbf{Z}/p^k)^{b_k},$$

   then the integers b_i are uniquely determined by G. [*Hint:* Consider the kernel of the homomorphism $f_i : G \to G$ that is multiplication by p^i. Show that f_1 and f_2 determine b_1. Proceed similarly.]
   (b) Let $p_1, \ldots, p_N$ be a sequence of distinct primes. Generalize (a) to a finite direct sum of terms of the form $(\mathbf{Z}/(p_i)^k)^{b_{ik}}$, where $b_{ik} \geq 0$.
   (c) Verify (c) of Theorem 4.3. That is, show that the betti number, invariant factors, and torsion coefficients of a finitely generated abelian group G are uniquely determined by G.
   (d) Show that the numbers t_i appearing in the conclusion of Theorem 4.2 are uniquely determined by F and R.

§5. HOMOLOGY GROUPS

Now we are ready to define the homology groups. First we must discuss the notion of "orientation."

Definition. Let σ be a simplex (either geometric or abstract). Define two orderings of its vertex set to be equivalent if they differ from one another by an even permutation. If $\dim \sigma > 0$, the orderings of the vertices of σ then fall into two equivalence classes. Each of these classes is called an **orientation** of σ. (If σ is a 0-simplex, then there is only one class and hence only one orientation of σ.) An **oriented simplex** is a simplex σ together with an orientation of σ.

If the points $v_0, \ldots, v_p$ are independent, we shall use the symbol

$$v_0 \ldots v_p$$

to denote the simplex they span, and we shall use the symbol

$$[v_0, \ldots, v_p]$$

to denote the oriented simplex consisting of the simplex $v_0 \ldots v_p$ and the equivalence class of the particular ordering $(v_0, \ldots, v_p)$.

*§11. THE COMPUTABILITY OF HOMOLOGY GROUPS

We have computed the homology groups of some familiar spaces, such as the sphere and the torus and the Klein bottle. Now we ask the question whether one can in fact compute homology groups in general. For finite complexes, the answer is affirmative. In this section, we present an explicit algorithm for carrying out the computation.

First, we prove a basic theorem giving a "normal form" for homomorphisms of finitely generated free abelian groups. The proof is constructive in nature. One corollary is the theorem about subgroups of free abelian groups that we stated earlier as Theorem 4.2. A second corollary is a theorem concerning standard bases for free chain complexes. And a third corollary gives our desired algorithm for computing the homology groups of a finite complex.

First, we need two lemmas with which you might already be familiar.

Lemma 11.1. *Let A be a free abelian group of rank n. If B is a subgroup of A, then B is free abelian of rank $r \leq n$.*

Proof. We may without loss of generality assume that B is a subgroup of the n-fold direct product $\mathbf{Z}^n = \mathbf{Z} \times \cdots \times \mathbf{Z}$. We construct a basis for B as follows:

Let $\pi_i : \mathbf{Z}^n \to \mathbf{Z}$ denote projection on the ith coordinate. For each $m \leq n$, let B_m be the subgroup of B defined by the equation

$$B_m = B \cap (\mathbf{Z}^m \times \mathbf{0}).$$

That is, B_m consists of all $\mathbf{x} \in B$ such that $\pi_i(\mathbf{x}) = 0$ for $i > m$. In particular, $B_n = B$. Now the homomorphism

$$\pi_m : B_m \to \mathbf{Z}$$

carries B_m onto a subgroup of $\mathbf{Z}$. If this subgroup is trivial, let $\mathbf{x}_m = \mathbf{0}$; otherwise, choose $\mathbf{x}_m \in B_m$ so that its image $\pi_m(\mathbf{x}_m)$ generates this subgroup. We assert that the non-zero elements of the set $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ form a basis for B.

First, we show that for each m, the elements $\mathbf{x}_1, \ldots, \mathbf{x}_m$ generate B_m. (Then, in particular, the elements $\mathbf{x}_1, \ldots, \mathbf{x}_n$ generate B.) It is trivial that $\mathbf{x}_1$ generates B_1; indeed if d is the integer $\pi_1(\mathbf{x}_1)$, then

$$\mathbf{x}_1 = (d, 0, \ldots, 0)$$

and B_1 consists of all multiples of this element.

Assume that $\mathbf{x}_1, \ldots, \mathbf{x}_{m-1}$ generate B_{m-1}; let $\mathbf{x} \in B_m$. Now $\pi_m(\mathbf{x}) = k\pi_m(\mathbf{x}_m)$ for some integer k. It follows that

$$\pi_m(\mathbf{x} - k\mathbf{x}_m) = 0,$$

so that $\mathbf{x} - k\mathbf{x}_m$ belongs to B_{m-1}. Then

$$\mathbf{x} - k\mathbf{x}_m = k_1\mathbf{x}_1 + \cdots + k_{m-1}\mathbf{x}_{m-1}$$

by the induction hypothesis. Hence $\mathbf{x}_1, \ldots, \mathbf{x}_m$ generate B_m.

Second, we show that for each m, the non-zero elements in the set $\{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$ are independent. The result is trivial when $m = 1$. Suppose it true for $m - 1$. Then we show that if

$$\lambda_1 \mathbf{x}_1 + \cdots + \lambda_m \mathbf{x}_m = \mathbf{0},$$

then it follows that for each i, $\lambda_i = 0$ whenever $\mathbf{x}_i \neq \mathbf{0}$; independence follows.

Applying the map π_m, we derive the equation

$$\lambda_m \pi_m(\mathbf{x}_m) = 0.$$

From this equation, it follows that either $\lambda_m = 0$ or $\mathbf{x}_m = \mathbf{0}$. For if $\lambda_m \neq 0$, then $\pi_m(\mathbf{x}_m) = 0$, whence the subgroup $\pi_m(B_m)$ is trivial and $\mathbf{x}_m = \mathbf{0}$ by definition. We conclude two things:

$$\lambda_m = 0 \quad \text{if} \quad \mathbf{x}_m \neq \mathbf{0},$$
$$\lambda_1 \mathbf{x}_1 + \cdots + \lambda_{m-1} \mathbf{x}_{m-1} = \mathbf{0}.$$

The induction hypothesis now applies to show that for $i < m$,

$$\lambda_i = 0 \quad \text{whenever} \quad \mathbf{x}_i \neq \mathbf{0}. \quad \square$$

For later use, we generalize this result to arbitrary free abelian groups:

Lemma 11.2. *If A is a free abelian group, any subgroup B of A is free.*

Proof. The proof given for the finite case generalizes, provided we assume that the basis for A is indexed by a well-ordered set J having a largest element. (And the well-ordering theorem, which is equivalent to the axiom of choice, tells us this assumption is justified.)

We begin by assuming A equals a direct sum of copies of $\mathbf{Z}$; that is, A equals the subgroup of the cartesian product $\mathbf{Z}^J$ consisting of all tuples $(n_\alpha)_{\alpha \in J}$ such that $n_\alpha = 0$ for all but finitely many α. Then we proceed as before.

Let B be a subgroup of A. Let B_β consist of those elements $\mathbf{x}$ of B such that $\pi_\alpha(\mathbf{x}) = 0$ for $\alpha > \beta$. Consider the subgroup $\pi_\beta(B_\beta)$ of $\mathbf{Z}$; if it is trivial define $\mathbf{x}_\beta = \mathbf{0}$, otherwise choose $\mathbf{x}_\beta \in B_\beta$ so $\pi_\beta(\mathbf{x}_\beta)$ generates the subgroup.

We show first that the set $\{\mathbf{x}_\alpha \mid \alpha \leq \beta\}$ generates B_β. This fact is trivial if β is the smallest element of J. We prove it in general by transfinite induction. Given $\mathbf{x} \in B_\beta$, we have

$$\pi_\beta(\mathbf{x}) = k\pi_\beta(\mathbf{x}_\beta)$$

for some integer k. Hence $\pi_\beta(\mathbf{x} - k\mathbf{x}_\beta) = 0$. Consider the set of those indices α for which $\pi_\alpha(\mathbf{x} - k\mathbf{x}_\beta) \neq 0$. (If there are none, $\mathbf{x} = k\mathbf{x}_\beta$ and we are through.) All of these indices are less than β, because $\mathbf{x}$ and $\mathbf{x}_\beta$ belong to B_β. Furthermore, this set of indices is *finite*, so it has a largest element γ, which is less than β. But this means that $\mathbf{x} - k\mathbf{x}_\beta$ belongs to B_γ, whence by the induction hypothesis, $\mathbf{x} - k\mathbf{x}_\beta$ can be written as a linear combination of elements $\mathbf{x}_\alpha$ with each $\alpha \leq \gamma$.

Second, we show that the non-zero elements in the set $\{\mathbf{x}_\alpha \mid \alpha \leq \beta\}$ are independent. Again, this fact is trivial if β is the smallest element of J. In general, suppose

$$\lambda_{\alpha_1}\mathbf{x}_{\alpha_1} + \cdots + \lambda_{\alpha_k}\mathbf{x}_{\alpha_k} + \lambda_\beta \mathbf{x}_\beta = \mathbf{0},$$

where $\alpha_i < \beta$. Applying π_β, we see that

$$\lambda_\beta \pi_\beta(\mathbf{x}_\beta) = 0.$$

As before, it follows that either $\lambda_\beta = 0$ or $\mathbf{x}_\beta = \mathbf{0}$. We conclude that

$$\lambda_\beta = 0 \quad \text{if} \quad \mathbf{x}_\beta \neq \mathbf{0},$$

and

$$\lambda_{\alpha_1}\mathbf{x}_{\alpha_1} + \cdots + \lambda_{\alpha_k}\mathbf{x}_{\alpha_k} = \mathbf{0}.$$

The induction hypothesis now implies that $\lambda_{\alpha_i} = 0$ whenever $\mathbf{x}_{\alpha_i} \neq \mathbf{0}$. $\square$

We now prove our basic theorem. First we need a definition.

Definition. Let G and G' be free abelian groups with bases $a_1, \ldots, a_n$ and $a'_1, \ldots, a'_m$, respectively. If $f: G \to G'$ is a homomorphism, then

$$f(a_j) = \sum_{i=1}^{m} \lambda_{ij} a'_i$$

for unique integers λ_{ij}. The matrix (λ_{ij}) is called the **matrix of** f relative to the given bases for G and G'.

Theorem 11.3. *Let G and G' be free abelian groups of ranks n and m, respectively; let $f: G \to G'$ be a homomorphism. Then there are bases for G and G' such that, relative to these bases, the matrix of f has the form*

$$B = \left[\begin{array}{ccc|c} b_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & b_l & \\ \hline & 0 & & 0 \end{array}\right]$$

where $b_i \geq 1$ and $b_1 \mid b_2 \mid \cdots \mid b_l$.

This matrix is in fact uniquely determined by f (although the bases involved are not). It is called a **normal form** for the matrix of f.

Proof. We begin by choosing bases in G and G' arbitrarily. Let A be the matrix of f relative to these bases. We shall give shortly a procedure for modifying these bases so as to bring the matrix into the normal form described. It is called "the reduction algorithm." The theorem follows. □

Consider the following "elementary row operations" on an integer matrix A:

(1) Exchange row i and row k.

(2) Multiply row i by -1.

(3) Replace row i by (row i) + q(row k), where q is an integer and $k \neq i$.

Each of these operations corresponds to a change of basis in G'. The first corresponds to an exchange of a_i' and a_k'. The second corresponds to replacing a_i' by $-a_i'$. And the third corresponds to replacing a_k' by $a_k' - qa_i'$, as you can readily check.

There are three similar "column operations" on A that correspond to changes of basis in G.

We now show how to apply these six operations to an arbitrary matrix A so as to reduce it to our desired normal form. We assume A is not the zero matrix, since in that case the result is trivial.

Before we begin, we note the following fact: If c is an integer that divides each entry of the matrix A, and if B is obtained from A by applying any one of these elementary operations, then c also divides each entry of B.

The reduction algorithm

Given a matrix $A = (a_{ij})$ of integers, not all zero, let $\alpha(A)$ denote the smallest non-zero element of the set of numbers $|a_{ij}|$. We call a_{ij} a **minimal entry** of A if $|a_{ij}| = \alpha(A)$.

The reduction procedure consists of two steps. The first brings the matrix to a form where $\alpha(A)$ is as small as possible. The second reduces the dimensions of the matrix involved.

Step 1. We seek to modify the matrix by elementary operations so as to *decrease* the value of the function α. We prove the following:

If the number $\alpha(A)$ fails to divide some entry of A, then it is possible to decrease the value of α by applying elementary operations to A; and conversely.

The converse is easy. If the number $\alpha(A)$ divides each entry of A, then it will divide each entry of any matrix B obtained by applying elementary operations to A. In this situation, it is not possible to reduce the value of α by applying elementary operations.

To prove the result itself, we suppose a_{ij} is a minimal entry of A that fails to divide some entry of A. If the entry a_{ij} fails to divide some entry a_{kj} in its *column*, then we perform a division, writing

$$\frac{a_{kj}}{a_{ij}} = q + \frac{r}{a_{ij}},$$

where $0 < |r| < |a_{ij}|$. Signs do not matter here; q and r may be either positive or negative. We then replace (row k) of A by (row k) $- q$(row i). The result is to replace the entry a_{kj} in the kth row and jth column of A by $a_{kj} - qa_{ij} = r$. The value of α for this new matrix is at most $|r|$, which is less than $\alpha(A)$.

A similar argument applies if a_{ij} fails to divide some entry in its *row.*

Finally, suppose a_{ij} divides each entry in its row and each entry in its column, but fails to divide the entry a_{st}, where $s \neq i$ and $t \neq j$. Consider the following four entries of A:

$$\begin{matrix} a_{ij} & \cdots & a_{it} \\ \vdots & & \vdots \\ a_{sj} & \cdots & a_{st} \end{matrix}$$

Because a_{ij} divides a_{sj}, we can by elementary operations bring the matrix to the form where the entries in these four places are as follows:

$$\begin{matrix} a_{ij} & \cdots & a_{it} \\ \vdots & & \vdots \\ 0 & \cdots & a_{st} + la_{it} \end{matrix}$$

If we then replace (row i) of this matrix by (row i) + (row s), we are back in the previous situation, where a_{ij} fails to divide some entry in its row.

Step 2. At the beginning of this step, we have a matrix A whose minimal entry divides every entry of A.

Apply elementary operations to bring a minimal entry of A to the upper left corner of the matrix and to make it positive. Because it divides all entries in its row and column, we can apply elementary operations to make all the other entries in its row and column into zeros. Note that at the end of this process, the entry in the upper left corner divides all entries of the matrix.

One now begins Step 1 again, applying it to the smaller matrix obtained by ignoring the first row and first column of our matrix.

Step 3. The algorithm terminates either when the smaller matrix is the zero matrix or when it disappears. At this point our matrix is in normal form. The only question is whether the diagonal entries $b_1, \ldots, b_l$ successively divide one another. But this is immediate. We just noted that at the end of the first application of Step 2, the entry b_1 in the upper left corner divides all entries of the matrix. This fact remains true as we continue to apply elementary operations. In particular, when the algorithm terminates, b_1 must divide each of $b_2, \ldots, b_l$.

A similar argument shows b_2 divides each of $b_3, \ldots, b_l$. And so on.

It now follows immediately from Exercise 4 of §4 that the numbers $b_1, \ldots, b_l$ are uniquely determined by the homomorphism f. For the number l of non-zero entries in the matrix is just the rank of the free abelian group $f(G) \subset G'$. And those numbers b_i that are greater than 1 are just the torsion coefficients $t_1, \ldots, t_k$ of the quotient group $G'/f(G)$.

Applications of the reduction algorithm

Now we prove the basic theorem concerning subgroups of free abelian groups, which we stated in §4.

Proof of Theorem 4.2. Given a free abelian group F of rank n, we know from Lemma 11.1 that any subgroup R is free of rank $r \leq n$. Consider the *inclusion* homomorphism $j : R \rightarrow F$, and choose bases $a_1, \ldots, a_r$ for R and $e_1, \ldots, e_n$ for F relative to which the matrix of j is in the normal form of the preceding theorem. Because j is a monomorphism, this normal form has no zero columns. Thus $j(a_i) = b_i e_i$ for $i = 1, \ldots, r$, where $b_i \geq 1$ and $b_1 \mid b_2 \mid \cdots \mid b_r$. Since $j(a_i) = a_i$, it follows that $b_1 e_1, \ldots, b_r e_r$ is a basis for R. □

Now we prove the "standard basis theorem" for free chain complexes.

Definition. A **chain complex** $\mathcal{C}$ is a sequence

$$\cdots \rightarrow C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \rightarrow \cdots$$

of abelian groups C_i and homomorphisms ∂_i, indexed with the integers, such that $\partial_p \circ \partial_{p+1} = 0$ for all p. The pth **homology group** of $\mathcal{C}$ is defined by the equation

$$H_p(\mathcal{C}) = \ker \partial_p / \operatorname{im} \partial_{p+1}.$$

If $H_p(\mathcal{C})$ is finitely generated, its betti number and torsion coefficients are called the betti number and torsion coefficients of $\mathcal{C}$ in dimension p.

Theorem 11.4 (Standard bases for free chain complexes). *Let $\{C_p, \partial_p\}$ be a chain complex; suppose each group C_p is free of finite rank. Then for each p there are subgroups U_p, V_p, W_p of C_p such that*

$$C_p = U_p \oplus V_p \oplus W_p,$$

where $\partial_p(U_p) \subset W_{p-1}$ and $\partial_p(V_p) = 0$ and $\partial_p(W_p) = 0$. Furthermore, there are bases for U_p and W_{p-1} relative to which $\partial_p : U_p \rightarrow W_{p-1}$ has a matrix of the form

$$B = \begin{bmatrix} b_1 & & 0 \\ & \ddots & \\ 0 & & b_l \end{bmatrix},$$

where $b_i \geq 1$ and $b_1 \mid b_2 \mid \cdots \mid b_l$.

Proof. Step 1. Let

$$Z_p = \ker \partial_p \qquad \text{and} \qquad B_p = \operatorname{im} \partial_{p+1}.$$

Let W_p consist of all elements c_p of C_p such that some non-zero multiple of c_p

belongs to B_p. It is a subgroup of C_p, and is called the group of **weak boundaries.** Clearly

$$B_p \subset W_p \subset Z_p \subset C_p.$$

(The second inclusion uses the fact that C_p is torsion-free, so that the equation $mc_p = \partial_{p+1} d_{p+1}$ implies that $\partial_p c_p = 0$.) We show that W_p is a direct summand in Z_p.

Consider the natural projection

$$Z_p \to H_p(\mathcal{C}) \to H_p(\mathcal{C})/T_p(\mathcal{C}),$$

where $T_p(\mathcal{C})$ is the torsion subgroup of $H_p(\mathcal{C})$. The kernel of this projection is W_p; therefore, $Z_p/W_p \simeq H_p/T_p$. The latter group is finitely generated and torsion-free, so it is free. If $c_1 + W_p, \ldots, c_k + W_p$ is a basis for Z_p/W_p, and $d_1, \ldots, d_l$ is a basis for W_p, then it is straightforward to check that $c_1, \ldots, c_k, d_1, \ldots, d_l$ is a basis for Z_p. Then $Z_p = V_p \oplus W_p$, where V_p is the group with basis $c_1, \ldots, c_k$.

Step 2. Suppose we choose bases $e_1, \ldots, e_n$ for C_p, and $e_1', \ldots, e_m'$ for C_{p-1}, relative to which the matrix of $\partial_p : C_p \to C_{p-1}$ has the normal form

$$\begin{array}{c} \\ e_1' \\ \vdots \\ e_l' \\ e_{l+1}' \\ \vdots \\ e_m' \end{array} \begin{array}{c} \begin{array}{cccc} e_1 \cdots e_l & & e_{l+1} \cdots e_n \end{array} \\ \left[\begin{array}{ccc|c} b_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & b_l & \\ \hline & & & \\ & 0 & & 0 \\ & & & \end{array}\right] \end{array}$$

where $b_i \geq 1$ and $b_1 \mid b_2 \mid \cdots \mid b_l$. Then the following hold:

(1) $e_{l+1}, \ldots, e_n$ is a basis for Z_p.

(2) $e_1', \ldots, e_l'$ is a basis for W_{p-1}.

(3) $b_1 e_1', \ldots, b_l e_l'$ is a basis for B_{p-1}.

We prove these results as follows: Let c_p be the general p-chain. We compute its boundary; if

$$c_p = \sum_{i=1}^{n} a_i e_i, \qquad \text{then} \qquad \partial_p c_p = \sum_{i=1}^{l} a_i b_i e_i'.$$

To prove (1), we note that since $b_i \neq 0$, the p-chain c_p is a cycle if and only if $a_i = 0$ for $i = 1, \ldots, l$. To prove (3), we note that any $p - 1$ boundary $\partial_p c_p$ lies in the group generated by $b_1 e_1', \ldots, b_l e_l'$; since $b_i \neq 0$, these elements are independent. Finally, we prove (2). Note first that each of $e_1', \ldots, e_l'$ belongs to W_{p-1}, since $b_i e_i' = \partial e_i$. Conversely, let

$$c_{p-1} = \sum_{i=1}^{m} d_i e_i'$$

be a $p - 1$ chain and suppose $c_{p-1} \in W_{p-1}$. Then c_{p-1} satisfies an equation of the form

$$\lambda c_{p-1} = \partial_p c_p = \sum_{i=1}^{l} a_i b_i e_i'$$

for some $\lambda \neq 0$. Equating coefficients, we see that $\lambda d_i = 0$ for $i > l$, whence $d_i = 0$ for $i > l$. Thus $e_1', \ldots, e_l'$ is a basis for W_{p-1}.

Step 3. We prove the theorem. Choose bases for C_p and C_{p-1} as in Step 2. Define U_p to be the group spanned by $e_1, \ldots, e_l$; then

$$C_p = U_p \oplus Z_p.$$

Using Step 1, choose V_p so that $Z_p = V_p \oplus W_p$. Then we have a decomposition of C_p such that $\partial_p(V_p) = 0$ and $\partial_p(W_p) = 0$. The existence of the desired bases for U_p and W_{p-1} follows from Step 2. □

Note that W_p and $Z_p = V_p \oplus W_p$ are uniquely determined subgroups of C_p. The subgroups U_p and V_p are not uniquely determined, however.

Theorem 11.5. *The homology groups of a finite complex K are effectively computable.*

Proof. By the preceding theorem, there is a decomposition

$$C_p(K) = U_p \oplus V_p \oplus W_p$$

where $Z_p = V_p \oplus W_p$ is the group of p-cycles and W_p is the group of weak p-boundaries. Now

$$H_p(K) = Z_p/B_p \simeq V_p \oplus (W_p/B_p) \simeq (Z_p/W_p) \oplus (W_p/B_p).$$

The group Z_p/W_p is free and the group W_p/B_p is a torsion group; computing $H_p(K)$ thus reduces to computing these two groups.

Let us choose bases for the chain groups $C_p(K)$ by orienting the simplices of K, once and for all. Then consider the matrix of the boundary homomorphism $\partial_p : C_p(K) \rightarrow C_{p-1}(K)$ relative to this choice of bases; the entries of this matrix will in fact have values in the set $\{0,1,-1\}$. Using the reduction algorithm described earlier, we reduce this matrix to normal form. Examining Step 2 of the preceding proof, we conclude from the results proved there the following facts about this normal form:

(1) The rank of Z_p equals the number of zero columns.

(2) The rank of W_{p-1} equals the number of non-zero rows.

(3) There is an isomorphism

$$W_{p-1}/B_{p-1} \simeq \mathbf{Z}/b_1 \oplus \mathbf{Z}/b_2 \oplus \cdots \oplus \mathbf{Z}/b_l.$$

Thus the normal form for the matrix of $\partial_p : C_p \to C_{p-1}$ gives us the torsion coefficients of K in dimension $p-1$; they are the entries of the matrix that are greater than 1. This normal form also gives us the rank of Z_p. On the other hand, the normal form for $\partial_{p+1} : C_{p+1} \to C_p$ gives us the rank of W_p. The difference of these numbers is the rank of Z_p/W_p—that is, the betti number of K in dimension p. □

EXERCISES

1. Show that the reduction algorithm is not needed if one wishes merely to compute the betti numbers of a finite complex K; instead all that is needed is an algorithm for determining the rank of a matrix. Specifically, show that if A_p is the matrix of $\partial_p : C_p(K) \to C_{p-1}(K)$ relative to some choice of basis, then

$$\beta_p(K) = \text{rank } C_p(K) - \text{rank } A_p - \text{rank } A_{p+1}.$$

2. Compute the homology groups of the quotient space indicated in Figure 11.1. [*Hint:* First check whether all the vertices are identified.]

3. Reduce to normal form the matrix

$$\begin{bmatrix} 2 & 6 & 4 \\ 4 & -7 & 4 \\ 4 & 8 & 4 \end{bmatrix}.$$

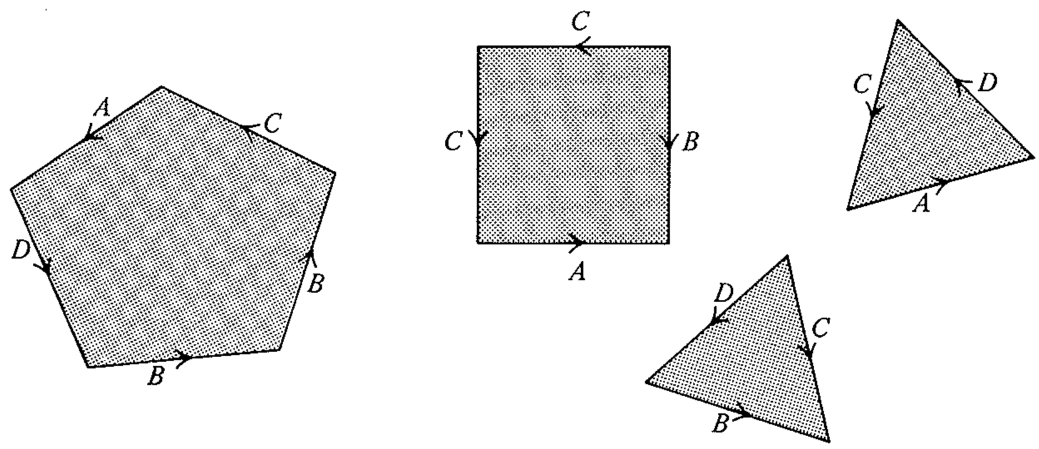

Figure 11.1