# HW 9 SOLUTIONS, MA525

## Problem 1

**(a):** First, the exact sequence of pairs:

$$\cdots \to H_n(S^n \setminus pt) \to H_n(S^n) \to H_n(S^n, S^n \setminus pt) \to H_{n-1}(S^n \setminus pt) \to \cdots$$

Since $S^n \setminus pt$ is contractible, the above sequence gives the isomorphism $H_n(S^n) \approx H_n(S^n, S^n \setminus pt)$. Excision of the sets $S^n \setminus U$ and $S^n \setminus V$ respectively, gives the isomorphisms: $H_n(S^n, S^n \setminus x) \approx H_n(U, U \setminus x)$ and $H_n(S^n, S^n \setminus y) \approx H_n(V, V \setminus y)$. Thus, fixing a generator $\alpha$ of $H_n(S^n)$ gives generators $\overline{\alpha}$ of $H_n(U, U \setminus x)$ and $\overline{\alpha}'$ of $H_n(V, V \setminus y)$.

Now consider the map $H_n(S^n) \to H_n(S^n, S^n \setminus f^{-1}y)$ in the long exact sequence of the pair $(S^n, S^n \setminus f^{-1}y)$. There is also the induced map $f_* : H_n(S^n, S^n \setminus f^{-1}y) \to H_n(S^n, S^n \setminus y)$. The induced map in homology for the restriction $f$ to the map of pairs: $(U, U \setminus x) \to (V, V \setminus y)$ is the composite $H_n(U, U \setminus x) \approx H_n(S^n, S^n \setminus x) \approx H_n(S^n) \to H_n(S^n, S^n \setminus f^{-1}y) \to H_n(S^n, S^n \setminus y) \to H_n(V, V \setminus y)$. So the local degree is independent of the choice of generator of $H_n(S^n)$.

**(b):** Suppose $U'$ and $V'$ are different open neighborhoods of $x$ and $y$ such that $f$ restricted to $U'$ is a homeomorphism with $f(U') \subset V'$. By Excision, there are isomorphisms $H_n(U, U \setminus x) \approx H_n(U \cup U', U \cup U' \setminus x)$ and $H_n(V, V \setminus y) \approx H_n(V \cup V', V \cup V' \setminus y)$, and the induced maps $f_*$ commutes with these isomorphisms. So, the local degree $deg_x f$ is independent of the choice of open sets.

**(c):** Taylor's theorem with remainders implies that there exists a positive constant $c$ and an open set $V$ around $x$, such that the remainder $f(v) - a(v)$ satisfies $\| f(v) - a(v) \| \leq c \| v - x \|^2$ for all $v \in V$. For the radius $r = \| Df \| / 2c$, where $\| Df \|$ is the norm of the derivative of $f$ at $x$, let $B(x, r)$ be the open ball centered at $x$ of radius $r$. Set $U = B(x, r) \cap V$. For all $v \in U$, we have the estimate for the remainder

$$\| f(v) - a(v) \| \leq c \| v - x \|^2 < \frac{\| Df \|}{2} \| v - x \|$$

By the triangle inequality, we get the estimate

$$\begin{aligned} \| a(v) - y \| &> \| Df \| \cdot \| v - x \| - c \| v - x \|^2 \\ &\geq \| Df \| \cdot \| v - x \| - c \left( \frac{\| Df \|}{2c} \right) \| v - x \| \\ &= \frac{\| Df \|}{2} \| v - x \| \end{aligned}$$

This implies that the straight line segment joining $f(v)$ and $a(v)$ is disjoint from $y$. Hence the straight line homotopy between $f(v)$ and $a(v)$ on $U$ is a homotopy of pairs $a, f : (U, U \setminus x) \to (\mathbb{R}^n, \mathbb{R}^n \setminus y)$.

This in turn shows that $deg_x f = deg_0 Df$.

**(d):** Consider the process of Guassian elimination on $T$ to get a diagonal matrix with $\pm 1$ on the diagonal. The desired elimination can be achieved by a sequence of row reductions of the form: $R_i \to aR_i + bR_j$ where $a > 0$. So it is enough to show that one can interpolate between an invertible matrix and the matrix after a basic row operation as above, through a family of invertible matrices. But this is obvious: just linear interpolate between $R_i$ and $aR_i + bR_j$. The resulting matrices remain invertible.

## Problem 2

**(a):** It suffices to compute $deg_\infty f$. There exists a radius $R$ large enough so that if $U$ is the set of complex numbers $z$ with $| z |> R$, then $f$ is homotopic to $z^n$ as a map of pairs: $(U, U \setminus \infty) \to (S^2, S^2 \setminus \infty)$. So $deg_\infty f = deg_\infty z^n = n$.

**(b):** Standard complex analysis shows that the number of roots of $f$ has to be finite (otherwise there is a convergent sequence, whose limit should also be a root. Then by the appropriate theorem from complex analysis $f$ ha to be a constant map). So we can choose a small enough neighborhood $V$ of 0 such that for distinct roots $w_1 \neq w_2$, the preimages of $V$ containing $w_1$ and $w_2$ are disjoint. Shrinking $V$ further if necessary, we can arrange that when $w$ is a root of multiplicity $k$, the number of pre-images of $z \neq 0 \in V$ in the neighborhood $U = f^{-1}V$ containing $w$ is $k$. Therefore, by part (a), adding the multiplicities of the roots gives $n$.

## Problem 3

Embed $S^n$ in $\mathbb{R}^{n+1}$ in the standard way, and let $p$ be the projecting in $\mathbb{R}^{n+1}$ to the first $n$ coordinates. The restriction of $p$ to $S^n$ maps it onto the unit ball $\mathbb{D}^n$ in $\mathbb{R}^n$. Let $q$ be the quotient map $\mathbb{D}^n \to \mathbb{D}^n/S^{n-1} = S^n$. The composition $q \circ p$ defines a map $S^n \to S^n$. The induced map on $H_n(S^n)$ factors through $H_n(\mathbb{D}^n) = 0$, and so has degree 0.

## Hatcher 2.2 Problem 11

Recall the cell structure of the space $X$. It has two 0-cells marked $u$ and $v$, four 1-cells marked $a, b, c, d$, three 2-cells marked $E, F, G$ and a single 3-cell given by the cube $I$ itself.

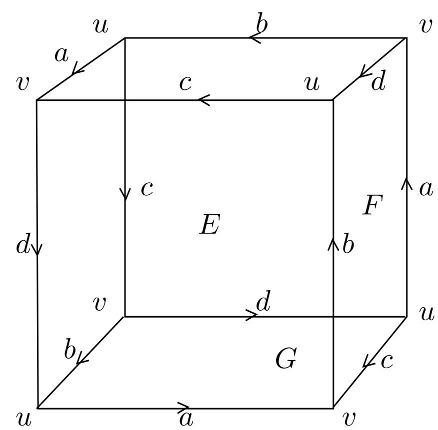


Figure 1. cell structure

We use Theorem 2.35 from Hatcher to compute the homology. First, we compute the cellular chain complex and the boundary maps.

$$\begin{aligned} C_4 &= 0 \\ C_3 &= \mathbb{Z}I \\ C_2 &= \mathbb{Z}E \oplus \mathbb{Z}F \oplus \mathbb{Z}G \\ C_1 &= \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c \oplus \mathbb{Z}d \\ C_0 &= \mathbb{Z}u \oplus \mathbb{Z}v \end{aligned}$$

The boundary maps are

$$
\begin{aligned}
\partial_4 &= 0 \\
\partial_3 I &= 0 \\
\partial_2 E &= a+b+c+d, \ \ \partial_2 F = a+d-b-c, \ \ \partial_2 G = a-c-d+b \\
\partial_1 a &= v-u, \ \ \partial_1 b = u-v, \ \ \partial_1 c = v-u, \ \ \partial_1 d = u-v
\end{aligned}
$$

The boundary map $\partial_3 I = 0$ because each pair of opposite faces, for instance the front and the back face, are equal to $\pm E$ (with opposite signs) after the gluing. Hence, $E$ cancels off with $-E$ in computing the boundary.

Finally, the homology computations:

$$
\begin{aligned}
H_3(X) &= Ker\partial_3/Im\partial_4 \approx \mathbb{Z} \\
Ker\partial_2 &= 0 \text{ by rank calculation on the matrix, } \ H_2(X) = 0 \\
Ker\partial_1 &= \mathbb{Z}(a+b) \oplus \mathbb{Z}(b+c) \oplus \mathbb{Z}(c+d) = \mathbb{Z}x_1 \oplus \mathbb{Z}x_2 \oplus \mathbb{Z}x_3, \\
Im\partial_2 &= Span(x_1+x_3, x_1-2x_2+x_3, x_1-x_3) = Span(x_1+x_3, 2x_2, 2x_1), \\
\text{So } H_1(X) &= Ker\partial_1/Im\partial_2 \approx \mathbb{Z}_2 \times \mathbb{Z}_2 \\
Im\partial_1 &= \mathbb{Z}(u-v), \ \text{ So } H_0(X) \approx \mathbb{Z}
\end{aligned}
$$